On Rings Whose Principal Ideals are Pure

Shaimaa H. Ahmad
shaima.hatem1977@gmail.com
College of Computer sciences and Mathematics
University of Mosul

Received on: 13/07/2005 **Accepted on: 09/10/2005**

ABSTRACT

In this work, we study rings whose every principal ideal is a right pure. We give some properties of right PIP – rings and the connection between such rings and division rings.

Keywords: Pure, division rings, reduced.

حول الحلقات التي فيها كل مثالي خاص نقي

شيماء حاتم أحمد
shaima.hatem1977@gmail.com
كلية علوم الحاسوب والرياضيات، جامعة الموصل

تاريخ استلام البحث: 2005/07/13 **تاريخ قبول البحث: 2005/10/09**

الملخص

في هذا العمل درسنا الحلقات التي يكون فيها كل مثالي خاص هو مثالي نقي أيمن. كما أعطينا بعض الخواص لهذه الحلقة اليمنى من النمط PIP ثم وضحنا العلاقة بينها وبين حلقات القسمة.

الكلمات المفتاحية: نقي، حلقات القسمة، مختزلة.

1. Introduction:

Throughout this paper , R will denote associative ring with identity . We recall that:

1) For any element a in R , we define the right annihilator of a by $r(a)=\{x \in R : ax=0\}$, and likewise the left annihilator $l(a)$.

2) Y , Z , J will denote respectively the right singular ideal , left singular ideal and Jacobson radical of R .

3) Following [2] a ring R is called reduced if R has no non – zero nilpotent elements .

4) R is called uniform if every non – zero ideal of R is essential, see [3].

2- PIP – Rings:

Following [1], an ideal I is said to be a right (left) pure if for every $a \in I$, there exists $b \in I$, such that $a = ab(ba)$.

Definition 2.1:

A ring R is said to be a right PIP – ring , if every principal ideal is pure .

Example:

The ring Z_6 is a PIP – ring.

Lemma 2.2:

Let R be a right PIP – ring. Then R is a reduced ring.

Proof:

Let $a \in R$, such that $a^2 = 0$. Since R is PIP – ring , then every principal ideal is pure and hence there exists $b \in aR$. Such $a = ab$, where $b = ar$, for some $r \in R$. Therefore $a = aar$, hence $a = a^2 r$, whence $a = 0$. Therefore R is a reduced ring .

Proposition 2.3:

Let R be PIP – ring . Then $Z(R) = 0$.

Proof:

Suppose that $Z(R) \neq 0$. Then by Lemma 7 [3] , there exists $0 \neq a \in Z(R)$. Such that $a^2 = 0$. Since R is a PIP – ring then $a = aar$, for some $r \in R$. In fact $Ra \cap l(ar) = 0$, since $a \in Z(R)$, then $l(ar)$ is essential left ideal . Therefore $Ra = 0$, where $a = 0$, a contradiction . Therefore $Z(R) = 0$.

Proposition 2.4:

Let R be PIP – ring, then $J(R) = 0$.

Proof:

Let $a \in J(R)$, since every principal ideal is pure then there exists $b = ar \in J(R)$ such that $a = aar$. For $u \in R$ such that $(1-b)u = 1$ then $a(1-b)u = a$ so $a = 0$ thus $J(R) = 0$.

3- The connection between PIP – rings and other rings:

In this section we study the connection between PIP – rings and division rings, semi ring – ring.

Theorem 3.1:

Let R be a right PIP – ring, without zero – divisors. Then R is a division ring.

Proof:

Let a be a non – zero element of R . Since R is PIP – ring , then every principal ideal is pure . There exists $b \in aR$, such that $a = ab$, where $b = ar$ for some $r \in R$. Therefore $a = aar$, whence $(1 - ar) \in r(a) = 0$, so $1 = ar$. Now since $r(a) = l(a)$ (Lemma 2 – 2) , so $a = ara$, gives $a(1 - ra) = 0$, so $1 = ra$.
Therefore a is invertible , whence R is a division ring .

The next result considers other conditions for a right PIP – ring to a division ring.

Theorem 3.2:

Let R be a right uniform, PIP – ring. Then R is a division ring.

Proof:

Let a be a non – zero element of R . As in (Theorem 3 – 1) . $a = aar$, for some $r \in R$. Since R is right uniform, so every right ideal is an essential ideal. In fact $Ra \cap l(ar) = 0$, let $x \in Ra \cap l(ar)$ implies that $x = sa \ and \ xar = 0$ for some $s \in R$, whence $saar = 0$ yielding $sa = 0$, so $x = 0$. Therefore $Ra \cap l(ar) = 0$ implies that $l(ar) = 0$, on the other hand, since $a = aar$ then $ra = raar$. If we set $e = ra$, note that e is an idempotent element of R, this implies that $ar = arar$,and hence $(1 - ar)ar = 0$, this implies that $1 - ar \in l(ar) = 0$. Therefore $ar = 1$ and hence a is a right inevitable. Now since $ar = 1$, we have $ara = a$, which implies thatSince R is reduced, we get $(1 - ra) \in r(a)$ · , so a is a $ra = 1$. Whence $1 - ra = 0$. Therefore $1 - ra \in r(a) = r(ar) = 0$ left inevitable. Hence R is a division ring.

Recall that a ring R is said to be semi prime if it has 0 as the only nilpotent ideal.

Proposition 3.3:

Let R be a right PIP – ring. Then R is a semi – prime ring.

Proof:

Let I be a non – zero right ideal of R such that $I^2 = 0$. Let $a \in I$ and R is PIP – ring, this implies that every principal ideal is pure, hence there exists $b \in aR$ such that $a = ab$ where $b = ar$, for some $r \in R$, therefore $a = aar \in I^2 = 0$, a contradiction . Therefore R is a semi – prime ring.

Theorem 3.4:

Let R be a right PIP – ring with $aR = Ra$, and without zero divisors then there exists a unit element $u \in R$, and idempotent element $e \in R$, such that $a = eu = ue$ and $a = (1-e)+u$.

Proof:

Let $0 \neq a \in R$ and since R is PIP – ring then $a = aar = a^2r$ and $a^2r = ra^2$ (without zero divisors). If we set $e = ar$, that e is an idempotent element of R, this implies that:

$a = ae = ea = ara$, let $u = a + e - 1$, then :

$$eu = e(a+e-1) = ea + e^2 - e = ea = a$$

$$ue = (a+e-1)e = ae + e^2 - e = ae = a$$

Since u is a unit, then there exists $v = ar + e - 1$

$$\begin{aligned} uv &= (a+e-1)(ar+e-1) = a^2r + a(e-1) + (e-1)ar + (e-1)^2 \\ &= e + ae - a + ear - ar + 1 - e \\ &= ar - ar + 1 = 1 \end{aligned}$$

and , $vu = (ar+e-1)(a+e-1) = 1$

Therefore $a = eu = ue$

Now, $(1-e) + u = 1 - e + a + e - 1 = a$

Thus $a = (1-e) + u$.

REFERENCES

[1] Al-Ezeh H. (1989) "Pure ideals in commutative reduced Gelfand rings with unity", **Arch. Math**. Vol.53, pp. 266 – 269.

[2] Yue Chi Ming (1983) "Maximal ideals in regular rings", Hokkaldo. **Math. Jour**. Vol.12, pp. 119 – 128.

[3] Yue Chi Ming (1983) "On quasi – injectivity and Von Neumann regularity", **Manatshefte, Math**. (95) , pp 25 – 32 .

[4] Yue Chi Ming (1986) "On semi prime and reduced ring", **Riv Math. Univ. Parma**. No.4, Vol. 12, pp.167 – 175.