Existence and Uniqueness of Positive Solution for Boundary Value Problem of Fractional Differential Equations

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ABSTRACT

The theory of Banach contraction and the fixed point theorems of Schaefer are used to explain the behavior of positive-solutions for the nonlinear differential equation of fractional order and the minimal values of form

$$^{c}D^{\gamma}\hbar(\upsilon) + g(\upsilon,\hbar(\upsilon)) = 0 \ , \quad \upsilon \in (0,1)$$

$$\hbar(0) = \hbar'(0) = \hbar''(0) = 0, \quad \hbar''(1) = a\hbar''(\zeta)$$

Where $3 < \gamma \leq 4$, $1 < a < \dfrac{1}{\zeta^{\gamma-3}}$, $0 < \zeta < 1$, $a \neq \dfrac{1}{\zeta^{\gamma-3}}$ and $^{c}D^{\gamma}$ is the standard Caputo fractional derivative .our main result is the presence and uniqueness of positive solution of the above equations. Finally, an example is given to illustrate our results.

Keywords: Positive solution, Fractional differential equations, Banach's and Schaefer's theorems.

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وجود ووحدانية الحل الموجب لمسألة القيم الحدودية لمعادلات تفاضلية كسرية

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الملخص

استخدمنا مبرهنة باناخ ونظرية شافير للنقطة الثابتة لإيجاد الحلول الموجبة لمسائل القيم الحدودية لمعادلة تفاضلية غير خطية ذات الرتب الكسرية. بالصيغة التالية

$$^{c}D^{\gamma}\hbar(\upsilon) + g(\upsilon,\hbar(\upsilon)) = 0 \ , \quad \upsilon \in (0,1)$$

$$\hbar(0) = \hbar'(0) = \hbar''(0), \quad \hbar''(1) = a\hbar''(\zeta)$$

حيث $3 < \gamma \leq 4, \ 1 < a < \dfrac{1}{\zeta^{\gamma-3}}, \ 0 < \zeta < 1, \ a \neq \dfrac{1}{\zeta^{\gamma-3}}$ و $^{c}D^{\gamma}$ يمثل الاشتقاق الكسري لـ Caputo.

تضمن عملنا دراسة وجود ووحدانية الحل الموجب لمنظومة المعادلات اعلاه ودعمنا بحثنا بمثال ليوضح نتائج البحث

الكلمات المفتاحية: الحل الموجب، معادلات تفاضلية ذات الرتب الكسرية، مبرهنتي باناخ وشافير.

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1- Introduction

Our results deal with the presence and uniqueness of positive solution for the fractional equations which is discussed below in this text.

$$^{c}D^{\gamma}\hbar(\upsilon) + g(\upsilon, \hbar(\upsilon)) = 0, \, \upsilon \in (0,1), \, \gamma \in (3,4] \tag{1}$$

Subject to the boundary conditions given by

$$\hbar(0) = \hbar'(0) = \hbar''(0) = 0, \quad \hbar''(1) = a\hbar''(\zeta) \tag{2}$$

Where $1 < a < \dfrac{1}{\zeta^{\gamma-3}}$, $0 < \zeta < 1$, and $^{c}D^{\gamma}$ is the standard Caputo fractional derivative and $g : [0,\infty) \rightarrow [0,\infty)$ is a continuous function.

In many areas, such as Physics- mechanics- chemistry and engineering, etc, fractional differential equations are found, see [6,8].

Wang, G. etal in [11] considered the problems of the following limit values:

$$^{c}D^{\alpha} Z(\upsilon) + b(\upsilon) f(Z(\theta(\upsilon))) = 0, \quad 0 < \upsilon < 1$$

$$Z(0) = Z''(0), \quad \beta Z(\eta) = Z(1)$$

Where $2 < \alpha \leq 3$, $0 < \eta < 1$, $0 < \beta < \dfrac{1}{\eta}$, using the well-known Guo-Krasnosel'skii fixed point theorem.

Also Chuanzhi Bai in [3] studied the following boundary value problems

$$D^{\alpha} Z(\upsilon) + f(\upsilon, Z_{\upsilon}) = 0, \quad \upsilon \in (0,1), \, \alpha \in (2,3]$$

$$Z'(0) = 0, \, Z'(1) = \lambdabar Z'(\eta)$$

And the initial condition $Z_0 = \Phi$ where $0 < \eta < 1$, $1 < \lambdabar < \dfrac{1}{\eta^{\alpha-2}}$.

And Cui etal [12], proved existence results via Schauder's theorem of the form

$$^{c}D^{\alpha} Z(\upsilon) + f(\upsilon, Z(\upsilon), {}^{c}D^{\beta}Z(\upsilon)) = 0, \, 0 < \upsilon < 1, \, 3 < \alpha \leq 4$$

$$Z(0) = Z'(0) = Z''(0) = 0, \, Z(1) = Z(\zeta), \, 0 < \zeta < 1$$

Recently, some papers on the existence and multiplicity of nonlinear Fractional differential equations (or positive solution) are currently being produced using nonlinear analysis techniques, see for example [4,2,1]. A positive solution of (1)-(2) means that g is positive at $0 < \upsilon < 1$ and meets the differential equation (1) and limit conditions (2).

It is structured as follows the rest of this article.

In section 2, we define some note, lemmas and definitions that are necessary. The results problem (1)-(2), based on Banach's contraction theorem and Schaefer's fixed point theorem, are presented in section 3.

2- Preliminaries

In this passage, we recall some of the definitions and lemmas necessary to define the problem solution (1)-(2).

Definition 2.1.[5] . The Caputo derivative of fractional order q is defined for q > 0 and for at least n-times constantly distinguishable function $I : [0,\infty) \rightarrow \Re$

$$^{c}D^{\ell} I(\upsilon) = \frac{1}{\Gamma(n-\ell)} \int_{0}^{\upsilon} (\upsilon-r)^{n-\ell-1} \, I^{(n)}(r) \, dr, \, n = [\ell]+1$$

Where $[\ell]$ is the integer part of ℓ.

Definition 2.2.[5] . Let $\delta > 0, let V : (0, \infty) \to \Re$ be continuous function, then the fractional integer of order δ is given by

$$I^{\delta} V(\gamma) = \frac{1}{\Gamma(\delta)} \int_0^{\gamma} (\gamma - s)^{\delta-1} V(s) ds$$

If the integeral is available.

Lemma 2.1. [9] For $\ell > 0$, the Fractional differential general solution of $^c D^{\ell} \lambda(\upsilon) = 0$, is given for

$\lambda(\upsilon) = a_1 \upsilon^{\ell-1} + a_2 \upsilon^{\ell-2} + ... + a_n \upsilon^{\ell-n}$ for some $a_i \in \Re, i = 1, 2, ..., n$ (n is the smallest integer such that $n \geq \ell$).

Lemma 2.2. Let $3 < \gamma \leq 4$ and let $g \in C([0,1], \Re)$ be a given function, then the limit value

$$^c D^{\gamma} \hbar(t) + g(t) = 0, \ t \in [0,1] \tag{3}$$

$$\hbar(0) = \hbar'(0) = \hbar''(0) = 0, \ \ \hbar''(1) = a\hbar''(\zeta) \tag{4}$$

Has a unique solution given by

$$\hbar(\upsilon) = \frac{-1}{\Gamma(\gamma)} \int_0^{\upsilon} (\upsilon - \sigma)^{\gamma-1} g(\sigma) d\sigma + \frac{\upsilon^{\gamma-1}}{1 - a\zeta^{\gamma-3}} \int_0^1 \frac{(1-\sigma)^{\gamma-3}}{\Gamma(\gamma)} g(\sigma) d\sigma$$

$$- \frac{a\upsilon^{\gamma-1}}{1 - a\zeta^{\gamma-3}} \int_0^{\zeta} \frac{(\zeta - \sigma)^{\gamma-3}}{\Gamma(\gamma)} g(\sigma) d\sigma \tag{5}$$

Proof: From Lemma 2.1, we may reduce (3) to an equivalent integral equation

$$\hbar(\upsilon) = \frac{1}{\Gamma(\gamma)} \int_0^{\upsilon} (\upsilon - \sigma)^{\gamma-1} g(\sigma) d\sigma + s_1 \upsilon^{\gamma-1} + s_2 \upsilon^{\gamma-2} + s_3 \upsilon^{\gamma-3} + s_4 \upsilon^{\gamma-4} \tag{6}$$

Where $s_i \in \Re$, by condition (4), we get $s_2 = s_3 = s_4 = 0$.

Now we shall find s_1

$$\hbar(\upsilon) = -\frac{1}{\Gamma(\gamma)} \int_0^{\upsilon} (\upsilon - \sigma)^{\gamma-1} g(\sigma) d\sigma + s_1 \upsilon^{\gamma-1}$$

$$\hbar'(\upsilon) = -\frac{1}{\Gamma(\gamma-1)} \int_0^{\upsilon} (\upsilon - \sigma)^{\gamma-2} g(\sigma) d\sigma + s_1(\gamma-1)\upsilon^{\gamma-2}$$

$$\hbar''(\upsilon) = -\frac{1}{\Gamma(\gamma-2)} \int_0^{\upsilon} (\upsilon - \sigma)^{\gamma-3} g(\sigma) d\sigma + s_1(\gamma-1)(\gamma-2)\upsilon^{\gamma-3}$$

$$\hbar''(1) = -\frac{1}{\Gamma(\gamma-2)} \int_0^1 (1-\sigma)^{\gamma-3} g(\sigma) d\sigma + s_1(\gamma-1)(\gamma-2)$$

$$a\hbar''(\zeta) = -\frac{a}{\Gamma(\gamma-2)} \int_0^{\zeta} (\zeta - \sigma)^{\gamma-3} g(\sigma) d\sigma + a s_1(\gamma-1)(\gamma-2)\zeta^{\gamma-3}$$

So the condition $\hbar''(1) = a\hbar''(\zeta)$ gives us

$$-\frac{1}{\Gamma(\gamma-2)}\int_0^1 (1-\sigma)^{\gamma-3} g(\sigma)d\sigma + s_1(\gamma-1)(\gamma-2) = -\frac{a}{\Gamma(\gamma-2)}\int_0^\zeta (\zeta-\sigma)^{\gamma-3} g(\sigma)d\sigma + a s_1(\gamma-1)(\gamma-2)\zeta^{\gamma-3}$$

By solving this equation we get

$$s_1 = \frac{1}{1-a\zeta^{\gamma-3}}\int_0^1 \frac{(1-\sigma)^{\gamma-3}}{\Gamma(\gamma)} g(\sigma)\,d\sigma - \frac{a}{1-a\zeta^{\gamma-3}}\int_0^\zeta (\zeta-\sigma)^{\gamma-3} g(\sigma)\,d\sigma$$

Hence the solution is

$$\hbar(\upsilon) = -\int_0^\upsilon \frac{(\upsilon-\sigma)^{\gamma-1}}{\Gamma(\gamma)} g(\sigma)\,d\sigma + \frac{\upsilon^{\gamma-1}}{1-a\zeta^{\gamma-3}}\int_0^1 \frac{(1-\sigma)^{\gamma-3}}{\Gamma(\gamma)} g(\sigma)\,d\sigma$$

$$-\frac{a\upsilon^{\gamma-1}}{1-a\zeta^{\gamma-3}}\int_0^\zeta \frac{(\zeta-\sigma)^{\gamma-3}}{\Gamma(\gamma)} g(\sigma)\,d\sigma \tag{7}$$

This completes the proof.

3- Existences Results

In this article, we discuss conditions for solutions to problems with the fraction limit value (1)-(2) using some certain fixed point theorems.

Let $C:[0,1]\to\Re$ be the Banach's area of all continuous functions with the norm $\|\hbar\| = \sup\{|\hbar(\upsilon)| : \upsilon\in[0,1]\}$ [7] .Define the operator $\wp : C \to C$ by

$$(\wp\hbar)(\upsilon) = -\int_0^\upsilon \frac{(\upsilon-s)^{\gamma-1}}{\Gamma(\gamma)} g(\sigma,\hbar(\sigma))d\sigma + \frac{\upsilon^{\gamma-1}}{(1-\zeta^{\gamma-3})}\int_0^1 \frac{(1-\sigma)^{\gamma-3}}{\Gamma(\gamma)} g(\sigma,\hbar(\sigma))d\sigma$$

$$-\frac{a\upsilon^{\gamma-1}}{(1-\zeta^{\gamma-3})}\int_0^\zeta \frac{(\zeta-\sigma)^{\gamma-3}}{\Gamma(\gamma)} g(\sigma,\hbar(\sigma))d\sigma \tag{8}$$

Of course, solutions to the fractional order boundary value problems (1)-(2) are the operator's fixed points.

The following well- known fixed-point theorems are needed to demonstrate our key results.

Theorem 3.1 [10]. (Banach contraction principle). Let (Q,d) be a fully unempty metric space with a mapping of a contract $P:Q\to Q$. Then P has a single point $\ell\in Q(i,e,\ P\ell=\ell)$.

Theorem 3.2 [10]. (Schaefer's fixed point theorem). Let H be a space Banach. Suppose that
$S:H\to H$ be an operator entirely continuous and the set $W=\{\lambda\in H: \lambda = \eta\,S\,\lambda, 0<\eta<1\}$ is constrained. Then S is in H with a fixed point.

Now-we are presenting the main results of this paper.

Theorem 3.3: Lipschits condition can be satisfied as a continuous function $g:[0,1]\times\Re\to\Re$

$(J1): |g(\upsilon,\hbar) - g(\upsilon,\lambda)| \le L|\hbar - \lambda|,\ L>0,\ \forall\upsilon\in[0,1],\ \hbar,\lambda\in\Re$

If

$$\frac{1}{\Gamma(\gamma+1)} + \frac{1+a\zeta^{\gamma-2}}{(\gamma-2)\Gamma(\gamma)(1-a\zeta^{\gamma-3})} = N < 1 \tag{9}$$

Then (1)-(2) has a unique solution on [0,1]

Proof:

Let $\sup\{|g(\upsilon,0)|,\upsilon\in[0,1]\}=M<\infty$, *and define*

$A_\rho=\{\hbar\in C:\|\hbar\|\leq\rho\}$, *where* $\rho>ML(1-LN)^{-1}$

Now we shall show that $\wp A_\rho\subset A_\rho$

From (J1) for $\hbar\in A_\rho$ *and* $\upsilon\in[0,1]$, we get

$$|g(\upsilon,\hbar(\upsilon))|\leq|g(\upsilon,\hbar(\upsilon))-g(\upsilon,0)|+|g(\upsilon,0)|$$

$$\leq L\|\hbar\|+M$$

$$\leq L\rho+M \tag{10}$$

Using (8) and (9), we have

$$\|\wp\hbar\|\leq\sup_{\upsilon\in[0,1]}\left\{-\int_0^\upsilon\frac{(\upsilon-\sigma)^{\gamma-1}}{\Gamma(\gamma)}|g(\sigma,\hbar(\sigma))|d\sigma+\frac{\upsilon^{\gamma-1}}{(1-a\zeta^{\gamma-3})}\int_0^1\frac{(1-\sigma)^{\gamma-3}}{\Gamma(\gamma)}|g(\sigma,\hbar(\sigma))|d\sigma\right.$$

$$\left.+-\frac{a\upsilon^{\gamma-1}}{(1-a\zeta^{\gamma-3})}\int_0^\zeta\frac{(\zeta-\sigma)^{\gamma-3}}{\Gamma(\gamma)}|g(\sigma,\hbar(\sigma))|d\sigma\right\}$$

$$\leq(L\rho+M)N\leq\rho$$

Thus $\wp A_\rho\subset A_\rho$. Now for $\hbar,\lambda\in C$,we get

$$\|(\wp\hbar)(\upsilon)-(\wp\lambda)(\upsilon)\|\leq\sup_{\upsilon\in[0,1]}\left\{-\int_0^\upsilon\frac{(\upsilon-\sigma)^{\gamma-1}}{\Gamma(\gamma)}d\sigma+\frac{\upsilon^{\gamma-1}}{(1-a\zeta^{\gamma-3})}\int_0^1\frac{(1-\sigma)^{\gamma-3}}{\Gamma(\gamma)}d\sigma\right.$$

$$\left.+\frac{a\upsilon^{\gamma-1}}{(1-a\zeta^{\gamma-3})}\int_0^\zeta\frac{(\zeta-\sigma)^{\gamma-3}}{\Gamma(\gamma)}d\sigma\right\}L\|\hbar-\lambda\|$$

$$\leq L\|\hbar-\lambda\|\sup_{\upsilon\in[0,1]}\left\{\frac{\upsilon^\gamma}{\Gamma(\gamma+1)}+\frac{\upsilon^{\gamma-1}(1+a\zeta^{\gamma-2})}{(\gamma-2)\Gamma(\gamma)(1-a\zeta^{\gamma-3})}\right\}$$

$$\leq L\|\hbar-\lambda\|\left[\frac{1}{\Gamma(\gamma+1)}+\frac{1+a\zeta^{\gamma-2}}{(\gamma-2)\Gamma(\gamma)(1-a\zeta^{\gamma-3})}\right]$$

$$\leq LN\|\hbar-\lambda\|$$

Since $LN<1,$ it follows that the operator $\wp$ is a contraction.

There is a single fixed point A_ρ for the operator $\wp$, which is the only solution of the problem (1)-(2) under the theorem 3.1 (Banach contraction principle), This is the proof. Our next outcome of life is based on the fixed point theorem of Schaefer's [10].

Theorem 3.4: The function $g:[0,1]\times\Re\to\Re$ is continuous function

(J2) A constant L>0 is available $|g(\upsilon,\hbar)|\leq L$ for all $\upsilon\in[0,1]$ *and* $\hbar\in\Re$.

There is at least one solution to the problem (1)-(2).

Proof: we show that $\wp$ defined by (9) has a fixed point by utilizing Schaefer's fixed point theorem. The proof will be given in several steps . Firstly, we show that $\wp$ is continuous.

Let $\{\hbar_n\}$ be a sequence such that $\hbar_n\to\hbar$, for each $\upsilon\in[0,1]$, we have

$$\|(\wp\hbar_n)(\upsilon) - (\wp\hbar)(\upsilon)\| = \sup_{\upsilon\in[0,1]} \left\| \left\{ \frac{-1}{\Gamma(\gamma)} \int_0^\upsilon (t-\sigma)^{\gamma-1}\big[g(\sigma,\hbar_n(\sigma)-g(\sigma,\hbar(\sigma)\big]\,d\sigma \right. \right.$$

$$+ \frac{\upsilon^{\gamma-1}}{\Gamma(\gamma)(1-a\zeta^{\gamma-3})} \int_0^1 (1-\sigma)^{\gamma-3}\big[g(\sigma,\hbar_n(\sigma)-g(\sigma,\hbar(\sigma)\big]\,d\sigma$$

$$\left. \left. - \frac{a\upsilon^{\gamma-1}}{\Gamma(\gamma)(1-a\zeta^{\gamma-3})} \int_0^\zeta (\zeta-\sigma)^{\gamma-3}\big[g(\sigma,\hbar_n(\sigma)-g(\sigma,\hbar(\sigma)\big]\,d\sigma \right\} \right\|$$

$$\leq \frac{1}{\Gamma(\gamma)} \int_0^\upsilon (\upsilon-\sigma)^{\gamma-1} \sup_{\upsilon\in[0,1]}\big|g(\sigma,\hbar_n(\sigma)-g(\sigma,\hbar(\sigma)\big|\,d\sigma$$

$$+ \frac{1}{\Gamma(\gamma)(1-a\zeta^{\gamma-3})} \int_0^1 (1-\sigma)^{\gamma-3} \sup_{\upsilon\in[0,1]}\big|g(\sigma,\hbar_n(\sigma)-g(\sigma,\hbar(\sigma)\big|\,d\sigma$$

$$+ \frac{a}{\Gamma(\gamma)(1-a\zeta^{\gamma-3})} \int_0^\zeta (\zeta-\sigma)^{\gamma-3} \sup_{\upsilon\in[0,1]}\big|g(\sigma,\hbar_n(\sigma)-g(\sigma,\hbar(\sigma)\big|\,d\sigma$$

$$\leq \|g(\sigma,\hbar_n(\sigma)-g(\sigma,\hbar(\sigma)\| \left[\frac{1}{\Gamma(\gamma+1)} + \frac{1+a\zeta^{\gamma-2}}{(\gamma-2)\Gamma(\gamma)(1-a\zeta^{\gamma-3})} \right]$$

$$\leq \|g(\sigma,\hbar_n(\sigma)-g(\sigma,\hbar(\sigma)\|N$$

Since g is continuous and $\hbar_n \to \hbar$, we have $\|\wp\hbar_n - \wp\hbar\| \to 0$ *as* $n \to \infty$. Therefore $\wp$ is continuous.

Secondly, we shall show that $\wp$ maps bounded sets into bounded sets in C. Let $A_\rho = \{\hbar \in C : \|\hbar\| \leq \rho\}$ be bounded set in C([0,1],R). Using condition (J2), we shall show $\wp$ is uniformly bounded on A_ρ.

$$\|\wp\hbar\| \leq \sup_{\upsilon\in[0,1]} \left\{ \frac{1}{\Gamma(\gamma)} \int_0^\upsilon (\upsilon-\sigma)^{\gamma-1}\big|g(\sigma,\hbar(\sigma))\big|\,d\sigma \right.$$

$$+ \frac{\upsilon^{\gamma-1}}{\Gamma(\gamma)(1-a\zeta^{\gamma-3})} \int_0^1 (1-\sigma)^{\gamma-3}\big|g(\sigma,\hbar(\sigma))\big|\,d\sigma$$

$$\left. + \frac{a\upsilon^{\gamma-1}}{\Gamma(\gamma)(1-a\zeta^{\gamma-3})} \int_0^\zeta (\zeta-\sigma)^{\gamma-3}\big|g(\sigma,\hbar(\sigma))\big|\,d\sigma \right\}$$

$$\leq L\left[\frac{1}{\Gamma(\gamma+1)} + \frac{1+a\upsilon^{\gamma-1}}{(\gamma-2)\Gamma(\gamma)(1-a\zeta^{\gamma-3})} \right]$$

$$\leq LN :\cong e\,,\ \hbar \in A_\rho,$$

Hence $\wp$ is uniformly bounded on A_ρ

Now for $\upsilon_1, \upsilon_2 \in [0,1]$, with $\upsilon_1 - \upsilon_2 < 0$, *and* $\hbar \in A_\rho$, we get

$$\left\| \wp \hbar(\upsilon_2) - \wp \hbar(\upsilon_1) \right\| \leq \frac{1}{\Gamma(\gamma)} \int_0^{\upsilon_1} \left[(\upsilon_2 - \sigma)^{\gamma-1} - (\upsilon_1 - \sigma)^{\gamma-1} \right] \left| g(\sigma, \hbar(\sigma)) \right| d\sigma$$

$$+ \frac{1}{\Gamma(\gamma)} \int_{\upsilon_1}^{\upsilon_2} (\upsilon_1 - \sigma)^{\gamma-1} \left| g(\sigma, \hbar(\sigma)) \right| d\sigma$$

$$+ \frac{\upsilon_2^{\gamma-1} - \upsilon_1^{\gamma-1}}{\Gamma(\gamma)(1 - a\zeta^{\gamma-3})} \int_0^1 (1 - \sigma)^{\gamma-3} \left| g(\sigma, \hbar(\sigma)) \right| d\sigma$$

$$+ \frac{a(\upsilon_2^{\gamma-1} - \upsilon_1^{\gamma-1})}{\Gamma(\gamma)(1 - a\zeta^{\gamma-3})} \int_0^\zeta (1 - \sigma)^{\gamma-3} \left| g(\sigma, \hbar(\sigma)) \right| d\sigma$$

$$\leq \frac{L}{\Gamma(\gamma)} \left\{ \int_0^{\upsilon_1} \left[(\upsilon_2 - \sigma)^{\gamma-1} - (\upsilon_1 - \sigma)^{\gamma-1} \right] d\sigma + \int_{\upsilon_1}^{\upsilon_2} (\upsilon_1 - \sigma)^{\gamma-1} d\sigma \right\}$$

$$+ \frac{L(\upsilon_2^{\gamma-1} - \upsilon_1^{\gamma-1})}{\Gamma(\gamma)(1 - a\zeta^{\gamma-3})} \int_0^1 (1 - \sigma)^{\gamma-3} d\sigma + \frac{aL(\upsilon_2^{\gamma-1} - \upsilon_1^{\gamma-1})}{\Gamma(\gamma)(1 - a\zeta^{\gamma-3})} \int_0^\zeta (\zeta - \sigma)^{\gamma-3} d\sigma$$

$$\leq \frac{L}{\Gamma(\gamma+1)} \left[2(\upsilon_2 - \upsilon_1)^\gamma + (\upsilon_2^\gamma - \upsilon_1^\gamma) \right] + \frac{L(\upsilon_2^{\gamma-1} - \upsilon_1^{\gamma-1})}{\Gamma(\gamma)(1 - a\zeta^{\gamma-3})(\gamma-2)} + \frac{aL(\upsilon_2^{\gamma-1} - \upsilon_1^{\gamma-1})\zeta^{\gamma-2}}{\Gamma(\gamma)(1 - a\zeta^{\gamma-3})(\gamma-2)}$$

$$\leq \frac{L}{\Gamma(\gamma+1)} \left[2(\upsilon_2 - \upsilon_1)^\gamma + (\upsilon_2^\gamma - \upsilon_1^\gamma) \right] + \frac{L(\upsilon_2^{\gamma-1} - \upsilon_1^{\gamma-1})(1 + a\zeta^{\gamma-2})}{\Gamma(\gamma)(1 - a\zeta^{\gamma-3})(\gamma-2)}$$

The right side of this inequality corresponds to zero   as $\upsilon_1 \to \upsilon_2$,   by   Arzela-Ascoli theorem, $\wp$ is completely continuous.

Finally, we need to show that the set $B = \{ \hbar \in C : \eta \wp R, \ 0 < \eta < 1 \}$ is-bounded. Let $\hbar \in B$ and $\upsilon \in [0,1]$, then

$$\hbar(\upsilon) = \eta \left\{ -\frac{1}{\Gamma(\gamma)} \int_0^\upsilon (\upsilon - \sigma)^{\gamma-1} g(\sigma, \hbar(\sigma)) d\sigma + \frac{\upsilon^{\gamma-1}}{\Gamma(\gamma)(1 - a\zeta^{\gamma-3})} \int_0^1 (1 - \sigma)^{\gamma-3} g(\sigma, \hbar(\sigma)) d\sigma \right.$$

$$\left. - \frac{a\upsilon^{\gamma-1}}{\Gamma(\gamma)(1 - a\zeta^{\gamma-3})} \int_0^\zeta (\zeta - \sigma)^{\gamma-3} g(\sigma, \hbar(\sigma)) d\sigma \right\}$$

Using $\eta < 1$, implies that

$$\left\| \hbar \right\| = \sup_{\upsilon \in [0,1]} \left\{ \left\| \eta (\wp \hbar)(\upsilon) \right\| \right\} < LN = M$$

Hence B is bounded. By Schaefer's theorem, we conclude that $\wp$ has a fixed point which is a solution of the fractional boundary value problems (1)-(2). The proof is complete.

4- Example

In this section we give example to illustrate the usefulness of our result.
Consider the nonlinear fractional differential equation:

$$^cD^{7/2} \hbar(\upsilon) = \frac{e^{-\upsilon} \left| \hbar(\upsilon) \right|}{(9 + e^\upsilon) \left| 1 + \hbar(\upsilon) \right|}, \quad \upsilon \in (0,1)$$

$$\hbar(0) = \hbar'(0) = \hbar''(0) = 0, \ \hbar''(1) = (1.2) \hbar''(0.2)$$

$$\left| g(\upsilon, \hbar(\upsilon)) - g(\upsilon, \lambda(\upsilon)) \right| \leq L \left| \hbar - \lambda \right|$$

$$\left| \frac{e^{-\upsilon} \left| \hbar(\upsilon) \right|}{(9 + e^{\upsilon}) \left| 1 + \hbar(\upsilon) \right|} - \frac{e^{-\upsilon} \left| \lambda(\upsilon) \right|}{(9 + e^{\upsilon}) \left| 1 + \lambda(\upsilon) \right|} \right| \leq \frac{e^{-\upsilon}}{(9 + e^{\upsilon})} \left(\frac{\left| \hbar - \lambda \right|}{\left| 1 + \hbar(\upsilon) \right| \left| 1 + \lambda(\upsilon) \right|} \right)$$

$$\leq \frac{e^{-\upsilon}}{(9 + e^{\upsilon})} \left| \hbar - \lambda \right|$$

$$\leq \frac{1}{8} \left| \hbar - \lambda \right|$$

Hence the condition (J1) holds with L=1/8.

We shall check that condition $\dfrac{1}{\Gamma(\gamma+1)} + \dfrac{1 + a\zeta^{\gamma-2}}{(\gamma-2)\Gamma(\gamma)(1 - a\zeta^{\gamma-3})} = N < 1$ is satisfied to

for appropriate values for $\gamma \in (3,4] = \dfrac{7}{2}$, *with* $a = 1.2$, $\zeta = 0.2$,

$$\frac{1}{\Gamma(\frac{7}{2}+1)} + \frac{1 + (1.2)(0.2)^{3/2}}{(\frac{7}{2}-2)\Gamma(\frac{7}{2})(1 - (1.2)(0.2)^{1/2})} = 0.5889 < 1.$$

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