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Family of One-Step A-Stable Optimized Third Derivative Hybrid Block Methods for Solving General Second-Order IVPs

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Abstract

This paper introduces three members of a one-step optimized third derivative hybrid block method family for solving general second-order initial value problems. The methodology incorporates optimization into the derivation process to achieve enhanced accuracy. Through rigorous analysis, it is demonstrated that all the derived methods are found to be zero-stable, consistent, A-stable, and convergent. The implementation of these newly derived methods is validated through numerical experimentation, where the results exhibit superior accuracy compared to certain existing numerical methods explored in the study. All the newly derived optimized third derivative hybrid block methods possessed very small error constants with high order accuracy.

Keywords:
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1 Introduction

In the realm of mathematics, numerical methods have garnered increasing interest among mathematicians. This is because everyday problems arising in fields such as science, engineering, and technology among others are often transformed into mathematical formulations. Many of these problems lack analytical solutions, thus necessitating the application of numerical techniques to approximate their solutions. However, given that numerical methods are essentially approximations of analytical solutions, it becomes paramount to exercise caution during their development to minimize the potential for significant errors Notably, numerous researchers have previously devised numerical methods for both specific and general second-order initial value problems (IVPs) such as [16-20] among others. For instance, Abdelrahim et al. [2] introduced a two-step optimized hybrid third derivative block method with a generalized one-off-step point. The optimal point was situated at the function (f) and its derivative counterpart (g) to develop an order $P = 8$ methods, rendering the method suitable for addressing general second-order IVPs. In the work of Shokri [3], a two-step explicit symmetric P-stable method was derived. This method incorporated the Obrechkoff and hybrid terms of orders four and six, specifically designed for solving second-order ordinary differential equations (ODEs). The method presented by Shokri represented an advancement over the approach proposed by Li and Wu [7].

Furthermore, in the work of Olabode and Omole [9], a continuous hybrid multistep method encompassing both one-step and two-step approaches was developed to address IVPs. The one-step method utilized three equally spaced off-step points, while the two-step method used two equally spaced off-step points [9]. The utilization of Legendre polynomials as basis functions facilitated the derivation of discrete schemes from the continuous framework. Additionally, Rufai and Ramos [15] developed an order $P = 7$, one-step hybrid block method that included a third derivative term, and this approach was formulated using three equally spaced off-step points at collocation, which

proved effective for solving problems such as Bratu's and Troesch's problems.

For a direct solution of IVPs, Abdelrahman and Omar [1] developed an order $P = 5$ one-step hybrid block method. This method utilized a power series polynomial as a basis function and incorporated three random off-step points to enhance its accuracy. In another approach, Olabode and Momoh [10] derived a two-step Chebyshev hybrid multistep method. This method utilized four equally spaced off-step points and the Chebyshev polynomial of the first kind as a basis function. The resulting method was tailored for the direct solution of second-order IVPs and BVPs.

In the realm of accuracy enhancement, Ramos and Singh [14] derived a two-step optimized third derivative hybrid block method of order $P = 7$. This method utilized two off-step points and was tailored for solving general second-order BVPs. Singla et al. [11] developed an optimized two-step hybrid block method of order $P = 5$. This method was implemented in a variable step size mode for the solution of IVPs. An optimized two-step hybrid block method that utilized two optimal points for the solution of IVPs was derived in [8,13]. Orakwelu et al. [12] developed an optimized two-step block hybrid method with four symmetric optimal points for the solution of IVPs.

Hybrid methods are highly efficient and have been proposed to circumvent the "Dahlquist zero-stability barrier" condition and to improve the accuracy of the block methods. Despite the hybrid block methods that were proposed by some of these authors, inefficiency in terms of accuracy for the solution of IVPs was discovered among others. Due to this, this study aims to improve on some of these setbacks.

The motivation behind this research is to develop a family of one-step optimized third-derivative hybrid block methods, utilizing various optimal points, to address general second-order IVPs. By incorporating optimization techniques into our methodology, our primary aim is to enhance the accuracy. The properties of the proposed methods shall be analyzed such as zero-stability, consistency, convergence, and linear stability. Numerical experiments shall be conducted on the proposed one-step optimized third derivative hybrid block methods. This contribution is expected to significantly advance the field of numerical techniques for effectively solving differential equations. This paper is organized as follows: Section 2 describes the derivation of the proposed methods. Section 3 contains an analysis of the properties of the derived methods. In section 4, the implementation is discussed and some numerical examples are presented. Section 5 presents the results and discussion. Finally, section 6 consists of a conclusion and future recommendations.

2 Derivation of the Optimized Hybrid Methods

This section outlines the procedure for deriving the members of the one-step optimized third derivative hybrid block methods designed for solving general second-order initial value problems, as given by

$$y''(x) = f(x, y, y'), y(x_0) = y_0, \quad y'(x_0) = y'_0. \qquad ...(1)$$

Here, $f \in \Re$ is a sufficiently differentiable function that adheres to a Lipschitz condition.
The main objective is to derive algorithms of the form given as

$$Y(x) = \mu_0(x)y_n + \mu_1(x)y_{n+1} + h^2[\sum_{j=0}^{1} \zeta_j(x)f_{n+j} + \sum_{i=1}^{m} \zeta_{p_i}(x)f_{n+p_i} + h\sum_{j=0}^{1} \xi_j(x)g_{n+j}], j = 0,1 \quad ...(2)$$

where m is the number of off-step points. The derivative of equation (2) is given in the form as

$$Y'(x) = \frac{1}{h}[\mu'_0(x)y_n + \mu'_1(x)y_{n+1} + h^2\sum_{j=0}^{1} \zeta'_j(x)f_{n+j} + h^2\sum_{i=1}^{m} \zeta'_{p_i}(x)f_{n+p_i} + h^3\sum_{j=0}^{1} \xi'_j(x)g_{n+j}] \quad ...(3)$$

to obtain additional equations. Let imposed that

$$Y'(x) = \delta(x). \quad ...(4)$$

To proceed with the derivation, let's denote the step size $h = x_{n+1} - x_n$ for $n = 0, 1, \ldots, N-1$ and approximate the exact solution $y(x)$ of equation (1) at the grid points $a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b$ in the interval $[a, b]$ by a polynomial of degree M given as

$$y(x) \approx Y(x) = \sum_{j=0}^{M} c_j (x - x_n)^j, \quad ...(5)$$

$$y''(x) \approx Y''(x) = \sum_{j=2}^{M} j(j-1)c_j (x-x_n)^{j-2}, \quad \ldots (6)$$

$$y'''(x) \approx Y'''(x) = \sum_{j=3}^{M} j(j-1)(j-2)c_j (x-x_n)^{j-3}, \ \ldots (7)$$

where c_j represents unknown coefficients to be determined, $M = t + u - 1$ when t is the interpolation point and u denotes the collocation points. These methods are derived by the introduction of specific off-step points defined as $x_{p_i} = x_n + p_i h$, where $0 < p_i < 1$ for $i = 1, 2, \ldots, m$. These off-step points are outlined in Table 1 where m is the number of off-step points.

Table 1: Distribution of the off-step points

m	**Off-Step Points**
1	r
2	$r, 1-r$
3	$r, \frac{1}{2}, 1-r$

In Case I, when $m = 1$, equation (5) is interpolated at $x = x_{n+j}$ for $j = 0, 1$ and collocate equation (6) at $x = x_{n+j}$ for $j = 0, r, 1$. Additionally, equation (7) is collocated at $x = x_{n+j}$ for $j = 0, 1$. This can be expressed as follows:

$$Y(x_{n+j}) = y_{n+j}, j = 0, 1,$$

$$Y''(x_{n+j}) = f_{n+j}, j = 0, r, 1,$$

$$Y'''(x_{n+j}) = g_{n+j}, j = 0, 1.$$

The above expression is solved using Mathematica to obtain the values of the unknown coefficients c_j, where $j = 0, 1, 2, \ldots, t + u - 1$. This process results in a system of $t + u$ equations with $t + u$ unknowns. Subsequently, we substitute the obtained solutions into equation (5) to yield the continuous approximation equation in the form of (2). Equation (2) is evaluated at $x = x_{n+r}$ and equation (3) at $x = x_{n+j}$ for $j = 0, r, 1$ to obtain the model equation, which is expressed as

$$y_{n+r} = \frac{h^2(r-1)(r-1(r-3(18+r(r-7(2r-3)))))f_n}{60r}$$

$$+\frac{(2h^2r^4 - 4h^2r^3 + h^2r^2 + h^2r + h^2)f_{n+r}}{60(r-1)r}$$

$$-\frac{h^2r(r(2r^4 - 7r^3 + r^2 + r + 10) - 6)f_{n+1}}{60(r-1)} +$$

$$\frac{(h^3r^7 - 5h^3r^6 + 9h^3r^5 - 5h^3r^4 - 3h^3r^3 + 4h^3r^2 - h^3r)g_n}{60(r-1)r} +$$

$$\frac{(h^3r^7 - 2h^3r^6 + 2h^3r^3 - h^3r^2)g_{n+1} + (-60r^3 + 120r^2 - 60r)y_n}{60(r-1)r} + ry_{n+1},$$

$$h\delta_n = -\frac{h^2(r(21r-2)-1)f_n}{60r^2} - \frac{h^2 f_{n+r}}{60(r-1)^2r^2} + \frac{h^2\big((16-9r)r-6\big)f_{n+1}}{60(r-1)^2} +$$

$$\frac{(-3h^3r^4 + 7h^3r^3 - 5h^3r^2 + h^3r)g_n + (2h^3r^4 - 3h^3r^3 + h^3r^2)g_{n+1}}{60(r-1)^2r^2}$$

$$+\frac{(-60r^4 + 120r^3 - 60r^2)y_n}{60(r-1)^2r^2} + y_{n+1},$$

$$h\delta_{n+r} = \frac{h^2(r(r(2r(r(3(r-2)r-5)+20)-21)+2)+1)f_n}{60r^2} +$$

$$\frac{(12h^2r^5 - 30h^4 + 20h^2r^3 - h^2)f_{n+r}}{60(r-1)^2r^2} - \frac{h^2(r(2(3(r-4)r+10)r^3 + 9r - 16) + 6)f_{n+1}}{60(r-1)^2}$$

$$+\frac{(3h^3r^8 - 16h^3r^7 + 33h^3r^6 - 30h^3r^5 + 7h^3r^4 + 7h^3r^3 - 5h^3r^2 + h^3r)g_n}{60(r-1)^2r^2}$$

$$+\frac{(3h^3r^8 - 8h^3r^7 + 5h^3r^6 + 2h^3r^4 - 3h^3r^3 + h^3r^2)g_{n+1}}{60(r-1)^2r^2}$$

$$+\frac{(-60r^4 + 120r^3 - 60r^2)y_n}{60(r-1)^2r^2} + y_{n+1},$$

$$h\delta_{n+1} = \frac{h^2(r(9r-2)-1)f_n}{60r^2} + \frac{h^2 f_{n+r}}{60(r-1)^2r^2} + \frac{h^2(r(21r-40)+18)f_{n+1}}{60(r-1)^2r^2}$$

$$+\frac{(2h^3r^4 - 5h^3r^3 + 4h^3r^2 - h^3r)g_n + (-3h^3r^4 + 5h^3r^3 - 2h^3r^2)g_{n+1}}{60(r-1)^2r^2}$$

$$+\frac{(-60r^4+120r^3-60r^2)y_n}{60(r-1)^2r^2} + y_{n+1}. \ldots (8)$$

To achieve maximum accuracy, we examine the continuous scheme $h\delta_{n+1}$ based on the model equation and optimize the local truncation error of $h\delta_{n+1}$ to determine the value of r. The local truncation errors are given by

$$L\left[y\left(x_{n+1}\right);h\right]=-\frac{\left(4-7r\right)y^{(7)}\left[x_n\right]h^7}{50400}+\frac{\left(3-2r\left(2+r\right)\right)y^{(8)}\left[x_n\right]h^8}{86400}+O\left(h^9\right). \ldots (9)$$

The coefficient of h^7 in equation (9) is set to zero to determine the value of r, which is given as

$$r=\frac{4}{7}. \quad \ldots(10)$$

Substituting the value of r into equation (9), the truncation error simplifies to

$$L\left[y\left(x_{n+1}\right);h\right]=\frac{y^{(8)}\left(x_n\right)h^8}{1411200}+O\left(h^9\right). \quad \ldots (11)$$

Therefore, the derived implicit hybrid block method is expressed as

$$y_{n+\frac{4}{7}}=-\frac{h^2\left(250965f_n+1124011f_{n+\frac{4}{7}}+4\left(26696f_{n+1}+h\left(7407g_n-636\right)g_{n+1}\right)\right)}{12101040}+\frac{3y_n}{7}+\frac{4y_{n+1}}{7},$$

$$h\delta_n=-\frac{h^2\left(2079f_n+2401f_{n+\frac{4}{7}}-160f_{n+1}+h(180g_n+48g_{n+1})\right)}{8640}-y_n+y_{n+1},$$

$$h\delta_{n+\frac{4}{7}}=\frac{h^2\left(1229121f_n+1523263f_{n+\frac{4}{7}}-1270624f_{n+1}+4h(40563g_n+41844g_{n+1})\right)}{864020744640}-y_n+y_{n+1},$$

$$h\delta_{n+1}=\frac{h^2(351f_n+2401f_{n+\frac{4}{7}}+1568f_{n+1}+h(36g_n-96g_{n+1}))}{8640}-y_n+y_{n+1}. \ldots (12)$$

In Case II, where $m = 2$, equation (5) is interpolated at $x = x_{n+j}$ for $j = 0, 1$, and equation (6) is collocated at $x = x_{n+j}$ for $j = 0, r, 1-r, 1$. Also, equation (7) is collocated at $x = x_{n+j}$ for $j = 0, 1$. Following procedures similar to those in Case I, equation (2) is evaluated at $x = x_{n+j}$ for $j = r, 1 - r, 1$, and equation (3) is evaluated at $x = x_{n+j}$ for $j = 0, r, 1 - r, 1$. The value of r is determined by optimizing the local truncation error of the continuous scheme $h\delta_{n+1}$. The local truncation errors are given as

$$\mathcal{L}[y(x_{n+1});h]=-\frac{(14(r-1)r+3)y^{(8)}[x_n]h^8}{604800}-\frac{(16+75(-1+r)r)y^{(9)}[x_n]h^9}{6350400}+0(h^{10}) \ldots (13)$$

To minimize the local truncation error, we set $14(r-1)r+3 = 0$ in equation (13). The unique solution within the range

$0 < r < 1 - r < 1$ is $r=\frac{1}{14}(7-\sqrt{7})$. … (14)

By substituting the value of r into equation (13), the truncation error becomes

$$L\left[y\left(x_{n+1}\right);h\right]=-\frac{y^{(9)}\left[x_n\right]h^9}{88905600}+O\left(h^{10}\right). \quad \ldots(15)$$

The following system of method equations is obtained

$$y_{n+\frac{1}{2}-\frac{1}{2\sqrt{7}}} = -\frac{h^2}{3457440}\Bigg(\left(3478\sqrt{7}+9604\right) f_n + 215992\, f_{n+\frac{1}{2}-\frac{1}{2\sqrt{7}}} + 13540\, f_{n+\frac{1}{14}(\sqrt{7}+7)} + (9604-3478\sqrt{7})\, f_{n+1} + 3h\left(\left(71\sqrt{7}-49\right) g_n + \left(71\sqrt{7}-49\right)\right) g_{n+1}\Big) + \frac{1}{14}\left(\left(7\sqrt{7}-7\right) y_n + \left(\sqrt{7}-49\right) y_{n+1}\right)$$

$$y_{n+\frac{1}{14}\left(7+\sqrt{7}\right)} = \frac{h^2}{3457440}\Bigg(\left(3478\sqrt{7}-9604\right) f_n - 135240\, f_{n+\frac{1}{2}-\frac{1}{2\sqrt{7}}} - 215992\, f_{n+\frac{1}{14}(\sqrt{7}+7)} - (9604+3478\sqrt{7})\, f_{n+1} + 3h\left(\left(49+71\sqrt{7}\right) g_n + \left(-49+71\sqrt{7}\right)\right) g_{n+1}\Big) + \frac{1}{14}\left(\left(7-\sqrt{7}\right) y_n + \left(7+\sqrt{7}\right) y_{n+1}\right),$$

$$h\delta_n = \frac{h^2}{540}\left(-71 f_n - 14\left(7+\sqrt{7}\right) f_{n+\frac{1}{2}-\frac{1}{2\sqrt{7}}} + 14\left(-7+\sqrt{7}\right) f_{n+\frac{1}{14}\left(7+\sqrt{7}\right)} - 3 f_{n+1} - 3hg_n\right) - y_n + y_{n+1},$$

$$h\delta_{n+\frac{1}{2}-\frac{1}{2\sqrt{7}}} = \frac{h^2}{105840}\Bigg(2\left(448\sqrt{7}+853\right) f_n - 3640\sqrt{7}\, f_{n+\frac{1}{2}-\frac{1}{2\sqrt{7}}} - 5712\sqrt{7}\, f_{n+\frac{1}{14}\left(7+\sqrt{7}\right)} + 2\left(-853+448\sqrt{7}\right) f_{n+1} + 3h\left(\left(37+34\sqrt{7}\right) g_n + \left(37-34\sqrt{7}\right) g_{n+1}\right)\Big) - y_n + y_{n+1},$$

$$h\delta_{n+\frac{1}{14}\left(\sqrt{7}+7\right)} = \frac{h^2}{105840}\Bigg(\left(1706-896\sqrt{7}\right) f_n + 5712\sqrt{7}\, f_{n+\frac{1}{2}-\frac{1}{2\sqrt{7}}} + 3640\sqrt{7}\, f_{n+\frac{1}{14}\left(7+\sqrt{7}\right)} - 2\left(853+448\sqrt{7}\right) f_{n+1} + 3h\left(\left(37-34\sqrt{7}\right) g_n + \left(37+34\sqrt{7}\right) g_{n+1}\right)\Big) - y_n + y_{n+1},$$

$$h\delta_{n+1} = \frac{h^2}{540}\Bigg(3 f_n - 14\left(-7+\sqrt{7}\right) f_{n+\frac{1}{2}-\frac{1}{2\sqrt{7}}} + 14\left(7+\sqrt{7}\right) f_{n+\frac{1}{14}\left(7+\sqrt{7}\right)} + 71 f_{n+1} - 3hg_{n+1}\Big) - y_n + y_{n+1}. \quad \ldots (16)$$

In Case III, where $m = 3$, equation (5) is interpolated at $x = x_{n+j}$ for $j = 0, \frac{1}{2}$, and equation (6) is collocated at $x = x_{n+j}$ for $j = 0, r, \frac{1}{2}, 1-r, 1$. Additionally, equation (7) is collocated at $x = x_{n+j}$ for $j = 0, 1$. The derivation follows a similar pattern as in Case I. Equation (2) is evaluated at $x = x_{n+j}$ for $j = r, 1-r, 1$ and equation (3) is evaluated at $x = x_{n+j}$ for $j = 0, r, \frac{1}{2}, 1-r, 1$.

This leads to the corresponding local truncation error, given by

$$L\left[y\left(x_{n+1}\right); h\right] = -\frac{\left(16(r-1)r+1\right) y^{(9)}\left[x_n\right] h^9}{25401600} - \frac{\left(7260(r-1)r+1217\right)\left[x_n\right] h^{10}}{52022476800} + O\left(h^{11}\right) \ldots (17)$$

To determine the value of r, let's set the coefficient of h^9 in equation (17) to zero. The unique solution within the range $0 < r < \frac{1}{2} < 1-r < 1$ is

$$r = \frac{1}{6}(3-\sqrt{3}) \qquad \ldots (18)$$

Substituting equation (18) into (17), the truncation error simplifies to

$$L\left[y\left(x_{n+1}\right);h\right] = -\frac{y^{(10)}\left(x_n\right)h^{10}}{7431782400} + O\left(h^{11}\right) \quad . \quad \ldots(19)$$

The block method equation can then be formulated in the following form

$$\begin{aligned} y_{n+\frac{1}{2}-\frac{1}{2\sqrt{3}}} = \frac{\left(\sqrt{3}-1\right)}{1451520}\Big(& -2h^2\Big(\left(992+603\sqrt{3}\right)f_n + 9\left(1448+665\sqrt{3}\right)f_{n+\frac{1}{2}-\frac{1}{2\sqrt{3}}} \\ & +8(1528-461\sqrt{3})f_{n+\frac{1}{2}} + 9\left(424-359\sqrt{3}\right)f_{n+\frac{1}{6}\left(\sqrt{3}+3\right)} + \\ & +\left(331\sqrt{3}+176\right)f_{n+1} + h^3\Big(\left(8-35\sqrt{3}\right)g_n + \left(40+51\sqrt{3}\right)g_{n+1}\Big)\Big) \\ & +\frac{1}{\sqrt{3}}y_n - \frac{1}{3}\left(\sqrt{3}-3\right)y_{n+\frac{1}{2}}, \end{aligned}$$

$$\begin{aligned} y_{n+\frac{1}{6}\left(3+\sqrt{3}\right)} = \frac{1}{1451520}\Big(& 2h^2\Big(\left(-817+389\sqrt{3}\right)f_n + 9\left(1501+783\sqrt{3}\right)f_{n+\frac{1}{2}-\frac{1}{2\sqrt{3}}} \\ & +8(2911+1989\sqrt{3})f_{n+\frac{1}{2}} + 9\left(-547+783\sqrt{3}\right)f_{n+\frac{1}{6}\left(3+\sqrt{3}\right)} \\ & -\left(817+155\sqrt{3}\right)f_{n+1} + h^3\Big(-\left(113+43\sqrt{3}\right)g_n + \left(113+11\sqrt{3}\right)g_{n+1}\Big)\Big) \\ & -\frac{1}{\sqrt{3}}y_n + \frac{1}{3}\left(\sqrt{3}+3\right)y_{n+\frac{1}{2}}, \end{aligned}$$

$$\begin{aligned} y_{n+1} = \frac{h^2}{26880}\Big(& 26f_n + 1566f_{n+\frac{1}{2}-\frac{1}{2\sqrt{3}}} + 3536f_{n+\frac{1}{2}} + 1566f_{n+\frac{1}{6}\left(3+\sqrt{3}\right)} \\ & +26f_{n+1} - 3h\left(g_n - g_{n+1}\right)\Big) - y_n + 2y_{n+\frac{1}{2}}, \end{aligned}$$

$$\begin{aligned} h\delta_n = \frac{h^2}{26880}\Big(& -2342f_n - 18\left(105+64\sqrt{3}\right)f_{n+\frac{1}{2}-\frac{1}{2\sqrt{3}}} - 560f_{n+\frac{1}{2}} \\ & +18\left(-105+64\sqrt{3}\right)f_{n+\frac{1}{6}(3+\sqrt{3})} - 38f_{n+1} - h\left(67g_n - 3g_{n+1}\right)\Big) - 2y_n + 2y_{n+\frac{1}{2}}, \end{aligned}$$

$$h\delta_{n+\frac{1}{2}-\frac{1}{2\sqrt{3}}} = \frac{1}{241920}\Bigg(2h^2\Big(\big(2213+704\sqrt{3}\big) f_n + \big(7047-6720\sqrt{3}\big) f_{n+\frac{1}{2}-\frac{1}{2\sqrt{3}}} + 8(1989-1472\sqrt{3}) f_{n+\frac{1}{2}} + 3\big(2349-1024\sqrt{3}\big) f_{n+\frac{1}{6}\left(3+\sqrt{3}\right)} + \big(-1979+704\sqrt{3}\big) f_{n+1}\Big) + h^3\Big(\big(245+96\sqrt{3}\big) g_n + \big(299-96\sqrt{3}\big) g_{n+1}\Big) - 2y_n + 2y_{n+\frac{1}{2}},$$

$$h\delta_{n+\frac{1}{2}} = \frac{h^2}{26880}\Bigg(-286 f_n + 18\big(87+76\sqrt{3}\big) f_{n+\frac{1}{2}-\frac{1}{2\sqrt{3}}} + 3536 f_{n+\frac{1}{2}} + 18\big(87-76\sqrt{3}\big) f_{n+\frac{1}{6}\left(3+\sqrt{3}\right)} + 338 f_{n+1} - h\big(35g_n + 29g_{n+1}\big)\Bigg) - 2y_n + 2y_{n+\frac{1}{2}},$$

$$h\delta_{n+\frac{1}{6}\left(3+\sqrt{3}\right)} = \frac{h^2}{241920}\Big(\big(4426-1408\sqrt{3}\big) f_n + 6\big(2349+1024\sqrt{3}\big) f_{n+\frac{1}{2}-\frac{1}{2\sqrt{3}}} + 16(1989+1472\sqrt{3}) f_{n+\frac{1}{2}} + 6\big(2349+2240\sqrt{3}\big) f_{n+\frac{1}{6}\left(3+\sqrt{3}\right)} - 2\big(1979+704\sqrt{3}\big) f_{n+1} + h\Big(\big(245-96\sqrt{3}\big) g_n + \big(299+96\sqrt{3}\big) g_{n+1}\Big)\Big) - 2y_n + 2y_{n+\frac{1}{2}},$$

$$h\delta_{n+1} = \frac{h^2}{26880}\Bigg(90 f_n + 18\big(279-64\sqrt{3}\big) f_{n+\frac{1}{2}-\frac{1}{2\sqrt{3}}} + 7632 f_{n+\frac{1}{2}} + 18\big(279+64\sqrt{3}\big) f_{n+\frac{1}{6}\left(3+\sqrt{3}\right)} + 2394 f_{n+1} - h\big(3g_n + 61g_{n+1}\big)\Big) \ldots (20) - 2y_n + 2y_{n+\frac{1}{2}}.$$

3 Analysis of the Methods

In this section, the properties of the one-step optimized third derivative hybrid block methods derived using the *m* off-step points are analyzed. Aspects such as order and error constants, zero stability, consistency, convergence, and linear stability have been considered. The derived methods are then reformulated into a matrix equation form, given as

$$A_1 Y_{n+1} = A_0 Y_n + hD_0\Delta_n + h^2(B_0F_n + B_1F_{n+1}) + h^3(C_0G_n + C_1G_{n+1}) \quad \ldots (21)$$

Here, $A_0, A_1, B_0, B_1, C_0, C_1$ *and* D_0 are matrices of coefficients, each with dimensions $m \times m$. Additionally, the vectors $Y_{n+1}, Y_n, F_n, F_{n+1}, G_n, G_{n+1}$ *and* Δ_n are defined as follows

$Y_n = (y_{n-p_1}, y_{n-p_2}, \ldots, y_{n-p_m}, y_n)^\tau$,

$Y_{n+1} = (y_{n+p_1}, y_{n+p_2}, \ldots, y_{n+p_m}, y_{n+1})^\tau$,

$F_n = (f_{n-p_1}, f_{n-p_2}, \ldots, f_{n-p_m}, f_n)^\tau$,

$F_{n+1} = (f_{n+p_1}, f_{n+p_2}, \ldots, f_{n+p_m}, f_{n+1})^\tau$,

$G_n = (g_{n-p_1}, g_{n-p_2}, \dots, g_{n-p_m}, g_n)^\tau$,

$G_{n+1} = (g_{n+p_1}, g_{n+p_2}, \dots, g_{n+p_m}, g_{n+1})^\tau$,
$\Delta_n = (\delta_{n-p_1}, \delta_{n-p_2}, \dots, \delta_{n-p_m}, \delta_n)^\tau$

3.1 Order and Error Constants

Let's consider the linear difference operator $\mathcal{L}$ associated with the developed optimized third-derivative hybrid block method, which is given by

$$\mathcal{L}[y(x_n); h] = \sum_{j=0}^{m+1}[\mu_j y(x_n + jh) - h\delta_j y'(x_n + jh) - h^2\zeta_j y''(x_n + jh) - h^3\xi_j y'''(x_n + jh)] \dots (22)$$

where $y(x_n)$ is a sufficiently differentiable function. By expanding Equation (22) in terms of $y(x_n + jh), y'(x_n + jh), y''(x_n + jh)$ *and* $y'''(x_n + jh)$ around x_n and collecting terms according to the powers of *h*, gives

$$\mathcal{L}[y(x_n); h] = C_0 y(x_n) + C_1 h y'(x_n) + C_2 h^2 y''(x_n) + \cdots + C_p h^p y^p(x_n) + \cdots \dots (23)$$

Here, C_j for $j = 0, 1, 2, \dots, N$ are vectors. A method is considered to be of order *p* if $C_0 = C_1 = C_2 = \cdots = C_{p+1} = 0$ *and* $C_{p+2} \neq 0$. The vector C_{p+2} is referred to as the error constant.

The order and error constants of the derived *m* off-step points methods are presented as follows

For Case I when *m* = 1, we have

$$C_7 = \left(\frac{2889}{1152960200}, \frac{1}{50400}, \frac{-1233}{94119200}, 0\right)^\tau$$

which indicates that the method has an order of *p* = 5.

For Case II when *m* = 2, we have

$$C_8 = \left(\frac{9}{172103680}, \frac{9}{172103680}, 0, \frac{3}{3073280\sqrt{7}}, \frac{-3}{3073280\sqrt{7}}, 0\right)^\tau$$

and this implies that the method has an order of *p* = 6.

In Case III, when *m* = 3, we obtain,

$$C_9 = \left(0, \frac{-1}{940584960\sqrt{3}}, \frac{1}{940584960\sqrt{3}}, 0, \frac{-1}{156764160}, \frac{1}{92897280}, \frac{-1}{156764160}, 0\right)^\tau.$$

This results in the method having an order of *p* = 7.

3.2 Zero Stability

Zero-stability is a critical property that determines the stability of a numerical method. It can be analyzed from equation (21) as the limit $h \to 0$. In this limit, equation (21) simplifies to

$$A_1 Y_{n+1} = A_0 Y_n. \dots (24)$$

The characteristic polynomial ρ(λ) is defined by

$$\rho(\lambda) = \det[\lambda A_1 - A_0]. \dots (25)$$

By calculating the characteristic polynomials for the derived methods with *m* = 1, 2, 3, we find; for *m* = 1: $\rho(\lambda) = \lambda(1 + \lambda)$, for *m* = 2: $\rho(\lambda) = \lambda^2(1 + \lambda)$ and for *m* = 3: $\rho(\lambda) = 2(-1 + \lambda)\lambda^3$. A numerical method is considered zero-stable if the roots λ_i, i = 1,2,...,*s* of the characteristic polynomial $\rho(\lambda)$ satisfy $|\lambda_i| \leq 1$. For those roots with $|\lambda_i| = 1$, their multiplicity must not exceed the order of the differential equation being solved (see Lawal et al. [6]). From the calculated characteristic polynomials, it's evident that all the derived methods satisfy the condition for zero stability.

3.3 Consistency

The Linear multistep method is considered consistent if it has an order $p \geq 1$ (see Lambert [5]). In this context, all the derived one-step optimized third derivative hybrid block methods have an order $p = m + 4 > 1$. Here, m represents the number of off-step points. Based on these results, it can be concluded that the proposed methods are indeed consistent.

3.4 Convergence

The convergence of the derived methods is determined by analyzing their consistency and zero-stability. This analysis is performed in accordance with Dahlquist's theorem 3.1, which provides the necessary conditions for the convergence of numerical methods for solving differential equations.

Theorem 3.1 Convergence

The necessary and sufficient conditions for the Linear multistep methods to be convergent are that they must be both consistent and zero-stable. Given that all the methods also meet the criteria for consistency and zero-stability, in accordance with Dahlquist's theorem 3.1, it can be concluded that these methods are convergent. This means that the methods provide accurate numerical approximation and converge to the true solution as the step size tends to zero.

3.5 Linear Stability

The stability region of a numerical method illustrates its behavior in a complex plane. It can be determined through the following approach. The Dahlquist [4] test equations $y' = \lambda y$, $y'' = \lambda^2 y$ and $y''' = \lambda^3 y$, where $\lambda \in$R, are applied on the matrix equation (21). By letting $z = \lambda h$ yield $Y_{n+1} = M(z)Y_n$, where $M(z) = (A_1 - z^2B_1 - z^3C_1)^{-1}.(A_0 + zD_0 + z^2B_0 + z^3C_0)$, is the amplification matrix. The stability of the method can be analyzed based on the eigenvalues of this matrix. For $m = 1$, the spectral radius is given by

$$\rho(z) = -\frac{332z^5 + 4389z^4 - 19065z^3 + 119760z^2 - 264600z + 264600}{2\left(76z^5 - 552z^4 + 735z^3 - 11170z^2 + 132300\right)},$$

For $m = 2$, the spectral radius is given by

$$\rho(z) = \frac{\left(501z^7 + 11172z^6 - 74924z^5 + 1080180z^4 - 5307120z^3 + 20618640z^2 - 44452800z + 44452800\right)}{3\left(39z^7 - 396z^6 + 560z^5 - 12132z^4 + 371280z^2 - 14817600\right)}$$

For $m = 3$, the spectral radius is given by

$$\rho(z) = -\frac{\left(19z^9 + 618z^8 - 7254z^7 + 88518z^6 - 483336z^5 + 1282968z^4 + 1118880z^3 - 57728160z^2 + 304819200z - 304819200\right)}{3\left(z^9 - 14z^8 - 4z^7 + 966z^6 - 1728z^5 - 29304z^4 - 594720z^2 + 101606400\right)}$$

For the methods derived with $m = 1, 2, 3$, the stability regions are characterized by the spectral radii of their amplification matrices. By analyzing the values of these spectral radii across different values of z (representing complex eigenvalues of the amplification matrix), one can determine the stability behavior and the region in which the method remains stable. This information provides insights into the method's suitability for solving stiff differential equations. The stability regions of these methods are shown in Fig. 1 to Fig. 3

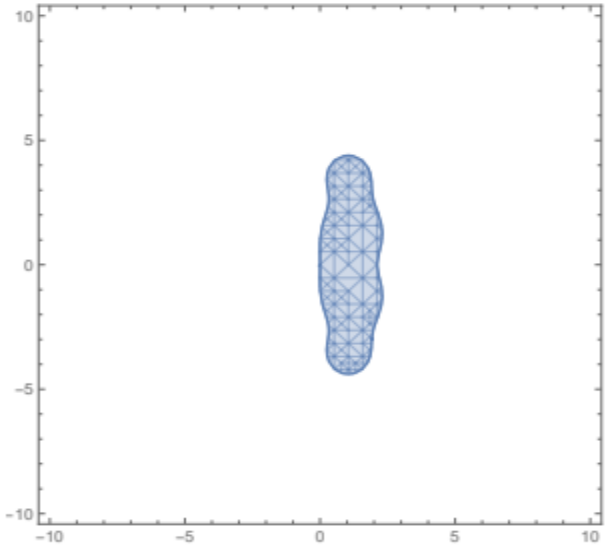

Fig. 1: Method *m* =1, Stability region

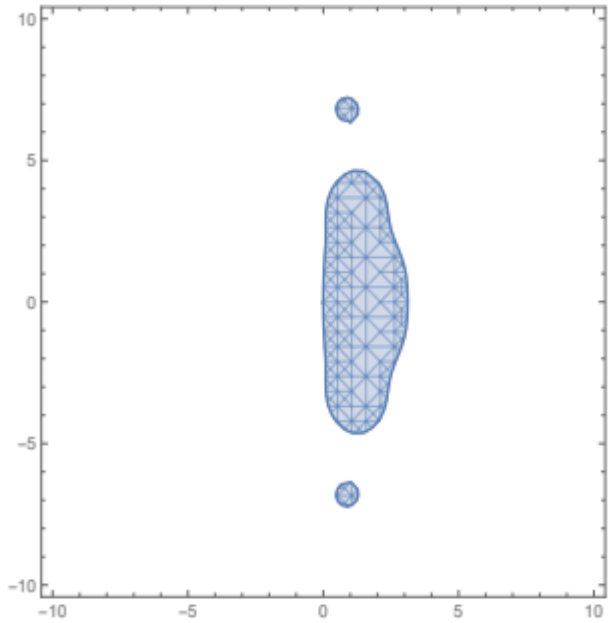

Fig.2: Method *m* =2, Stability region

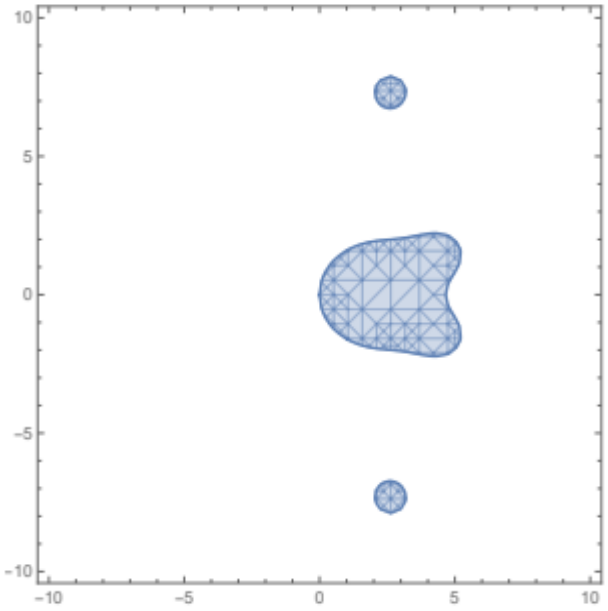

Fig. 3: Method *m* =3, Stability region

A numerical method is said to be *A*-stable if its region of absolute stability contains the entire negative (left) complex half-plane *C* (see Lambert [5]). Based on the above analysis, all methods derived with *m* = 1,2,3 exhibit stability regions entirely contained within the left half-plane of the complex plane, indicating that these methods are *A*- stable.

4 Implementation of Derived Methods

The implementation process of the proposed methods is explained. The methods are effectively implemented as one-step block numerical integrators for solving (1) and simultaneously obtaining the approximations $(y_{n+r}, \dots, y_{n+1})^{\tau}$, with n ranging from 0 to $N-1$, over non-overlapping subintervals $[x_0, x_1], \dots, [x_{N-1}, x_N]$.

Step 1: Set N and $h = \dfrac{(b-a)}{N}$, where h represents a constant step size and $N > 0$ is the partition integer. For $n = 0$, the values of $(y_r, \dots, y_1)^{\tau}$ are simultaneously determined over the interval $[x_0, x_1]$ using the known value y_0 from the initial value problem (1).

Step 2: For $n = 1$, the values of $(y_{1+r}, \dots, y_2)^{\tau}$ are simultaneously obtained over the interval $[x_1, x_2]$, with y_1 being known from the previous block.

Step 3: The process continues for $n = 2, 3, \dots, N-1$, obtaining approximate solutions for equation (1) over sub-intervals $[x_2, x_3], \dots, [x_{N-1}, x_N]$, given that these sub-intervals do not overlap.

The derivations, analysis, and implementations of these methods were carried out using the Mathematica 13.0 edition programming language. Nonlinear problems were solved using the Find Root command, while linear problems were solved using the NSolve command in Mathematica.

Problem 1

Consider the following non-linear stiff systems that have been solved, among others, by authors such as [1,9]

$$y_1'' = \frac{-y_1}{\sqrt{y_1^2 + y_2^2}}, \quad y_1(0) = 1, \quad y_1'(0) = 0,$$

$$y_2'' = \frac{-y_2}{\sqrt{y_1^2 + y_2^2}}, \quad y_2(0) = 0, \quad y_2'(0) = 1,$$

The exact solution is given as $y_1(x) = cos(x)$ and $y_2(x) = sin(x)$.

Problem 2

Consider the non-linear stiff problem solved by [2,20]

$y'' = 2y^3, y(1) = 1, y'(1) = -1.$

The exact solution is given as $y(x) = \frac{1}{x}$.

Problem 3

Consider the linear stiff problem solved by [18]

$y'' = -\lambda^2 y, y(0) = 1, y'(0) = 2, \lambda = 2.$

The exact solution is given as $y(x) = \cos(2x) + \sin(2x)$.

Problem 4

Consider the non-linear IVP solved by [13]

$y'' = 50y^3, y(1) = \frac{1}{6}, y'(1) = -\frac{5}{36}.$

The exact solution is given as $y(x) = \frac{1}{(1+5x)}$.

The problem is solved with step sizes of $h = 0.1$ and the true value at $x = 1 + h$ is used as the second starting value.

Problem 5

Consider the linear IVP solved by [16]

$y'' = 8y' - 17y, y(0) = -4, y'(0) = -1.$

The exact solution is given as $y(x) = -4e^{4x}cos(x) + 15e^{4x}sin(x)$.

Problem 6

Consider the systems of linear stiff IVP

$$y_1'' = (\in -2)y_1 + (2 \in -2)y_2, \; y_1(0) = 2, \; y_1'(0) = 0$$

$$y_2'' = (1- \in)y_1 + (1-2 \in)y_2, \; y_2(0) = -1, \; y_2'(0) = 0$$

with the exact solution $y_1(x) = 2\ cos(x)$,

$y_2(x) = -cos(x)$ and $\epsilon = 2500$.

5 Results and Discussion

The numerical results obtained from applying the proposed methods to selected problems that have been previously used in published studies for numerical experimentation are presented. The aim is to demonstrate the enhanced accuracy of the proposed methods.

Note: The new methods derived when $m = 1,2,3$ are also denoted as NMm1, NMm2, and NMm3 respectively.

Table 2: Comparison of absolute y_1 error for problem 1

x	Error y_1 in [1] $p = 5$ $h = 0.01$	Error y_1 in NMm1 $p = 5$ $h = 0.1$	Error y_1 in NMm2 $p = 6$ $h = 0.1$	Error y_1 in NMm3 $p = 7$ $h = 0.1$	Error y_1 in [10] $p = 7$ $h = 0.1$
0.2	4.2086×10^{-11}	4.6396×10^{-13}	1.1100×10^{-16}	0.0000	1.2700×10^{-16}
0.4	2.9457×10^{-10}	2.7693×10^{-12}	2.2200×10^{-16}	1.1100×10^{-16}	8.8900×10^{-16}
0.6	7.5965×10^{-10}	8.1538×10^{-12}	3.3310×10^{-16}	5.5510×10^{-16}	1.9300×10^{-16}
0.8	1.1786×10^{-9}	1.7437×10^{-11}	6.6610×10^{-16}	8.8820×10^{-16}	3.2800×10^{-15}
1	1.1803×10^{-9}	3.0995×10^{-11}	8.8820×10^{-16}	1.2212×10^{-15}	4.8000×10^{-15}

Table 3: Comparison of absolute y_2 error for problem 1

x	Error y_2 in [1] $p = 5$ $h = 0.01$	Error y_2 in NMm1 $p = 5$ $h = 0.1$	Error y_2 in NMm2 $p = 6$ $h = 0.1$	Error y_2 in NMm3 $p = 7$ $h = 0.1$	Error y_2 in [10] $p = 7$ $h = 0.1$
0.2	3.1869×10^{-10}	5.9011×10^{-12}	2.7800×10^{-17}	3.0000×10^{-17}	2.5900×10^{-15}
0.4	1.0828×10^{-9}	1.9233×10^{-11}	5.5500×10^{-17}	1.1100×10^{-16}	4.9800×10^{-15}
0.6	1.7732×10^{-9}	3.9016×10^{-11}	0.0000	2.2200×10^{-16}	6.9900×10^{-15}
0.8	1.8627×10^{-9}	6.3924×10^{-11}	2.2200×10^{-16}	0.0000	8.4800×10^{-15}
1	1.0816×10^{-9}	9.2521×10^{-11}	2.2200×10^{-16}	0.0000	9.3000×10^{-15}

The numerical results in Tables 2 and 3 demonstrate that the one-step optimized hybrid block method with m = 1 is more accurate than the one-step hybrid block method with three off-step points derived by Abdelrahim and Omar [1]. It's worth noting that both the new methods m = 2,3 have better accuracy than the two-step hybrid block method in [10] which was developed with four equidistant off-step points

Table 4: Comparison of absolute errors for problem 2 using $h = 0.1$

x	Error in [20] $p = 5$	Error in NMm1 $p = 5$	Error in NMm2 $p = 6$	Error in NMm3 $p = 7$	Error in [2] $p = 8$
1.1	5.0300×10^{-7}	7.0534×10^{-9}	2.9766×10^{-12}	3.4417×10^{-15}	1.6603×10^{-10}
1.2	1.3900×10^{-6}	1.8007×10^{-8}	5.1502×10^{-12}	5.7732×10^{-15}	8.2396×10^{-10}
1.3	1.9720×10^{-6}	3.1511×10^{-8}	7.1743×10^{-12}	7.7716×10^{-15}	6.8322×10^{-7}
1.4	1.9670×10^{-4}	4.7097×10^{-8}	9.3027×10^{-12}	9.9920×10^{-15}	1.3895×10^{-6}
1.5			1.1649×10^{-11}	1.2324×10^{-14}	2.2801×10^{-6}

In Table 4, the new method derived when m = 1 exhibited better performance by enhancing the accuracy of the solution for the stiff problem compared to the three-step block method in Yakubu et al. [20] with only one off-step point at collocation. The newly derived methods with m = 2 and m = 3 have outperformed the two-step third derivative hybrid block method in [2] which was derived with one off-step point at both the second and third derivative terms.

Table 5: Comparison of absolute errors for problem 3 using $h = 0.01$

x	Error in [18] $p = 5$	Error in NMm1 $p = 5$	Error in NMm2 $p = 6$	Error in NMm3 $p = 7$
0.1	3.4090×10^{-11}	2.2200×10^{-16}	2.2200×10^{-16}	2.2200×10^{-16}
0.2	3.2390×10^{-11}	6.6610×10^{-16}	1.9260×10^{-16}	1.5490×10^{-16}
0.3	3.4650×10^{-11}	1.1102×10^{-15}	1.8390×10^{-16}	1.0840×10^{-16}
0.4	2.4000×10^{-13}	1.5543×10^{-15}	3.1940×10^{-16}	2.0620×10^{-16}
0.5	1.7800×10^{-12}	1.9984×10^{-15}	2.5500×10^{-16}	1.0410×10^{-16}
0.6	7.4670×10^{-11}	2.6645×10^{-15}	2.0800×10^{-16}	1.9600×10^{-17}
0.7	3.9040×10^{-11}	3.3307×10^{-15}	2.8630×10^{-16}	6.0400×10^{-17}
0.8	4.1320×10^{-11}	3.9968×10^{-15}	2.3650×10^{-16}	2.6900×10^{-17}
0.9	1.1970×10^{-10}	4.4409×10^{-15}	3.5600×10^{-16}	5.5300×10^{-17}
1	8.3420×10^{-11}	5.1070×10^{-15}	2.5750×10^{-16}	8.0400×10^{-17}

Table 5 presents a comparison of absolute errors between the new method with m = 1 and the method proposed by Omole and Ogunware [18]. The table reveals a significant difference in the accuracy of the new method with m = 1 compared to the 3-step method with 2 off-step points. It is evident from Table 5 that the new method exhibits superior accuracy. The newly developed methods with m = 2 and m = 3 exhibit superior accuracy compared to the 3-step method presented in [18], which was derived with only 2 off-step points. This highlights the effectiveness of the additional off-step points incorporated into our methods and emphasizes their potential for achieving more accurate solutions to differential equations.

Table 6: Comparison of absolute errors for problem 4 using $h = 0.1$

x	**Error in [3]** $P = 6$	**Error in NMm1** $P = 5$	**Error in NMm2** $P = 6$	**Error in NMm3** $P = 7$
5	1.1325×10^{-5}	3.2701×10^{-9}	3.8123×10^{-13}	1.8040×10^{-16}
10	4.2034×10^{-5}	9.1569×10^{-9}	8.5432×10^{-13}	1.9430×10^{-16}
15	6.1478×10^{-4}	1.7965×10^{-8}	1.5879×10^{-12}	1.3880×10^{-16}
20	9.0336×10^{-4}	3.0482×10^{-8}	2.6479×10^{-12}	1.0410×10^{-16}

Table 6 presents a comparison of the new methods derived with m = 1,2,3 and the method developed by Shokri [3]. The results in Table 6 demonstrate that the new methods outperform the compared method in terms of accuracy.

Table 7: Comparison of absolute errors for problem 5 using $h = 0.01$

x	**Error in NMm1** $p = 5$	**Error in NMm2** $p = 6$	**Error of BhyNM3 in [16]** $p = 6$	**Error in NMm3** $p = 7$
0.1	6.5725×10^{-14}	0.0000	2.1000×10^{-12}	8.8820×10^{-16}
0.2	2.0490×10^{-13}	1.7764×10^{-15}	4.7000×10^{-12}	9.2440×10^{-16}
0.3	4.2869×10^{-13}	3.5527×10^{-15}	7.1000×10^{-12}	2.6146×10^{-15}
0.4	7.4299×10^{-13}	7.5495×10^{-15}	9.6000×10^{-12}	2.3313×10^{-15}
0.5	1.1620×10^{-13}	9.7700×10^{-15}	1.2200×10^{-11}	4.0373×10^{-15}
0.6	1.6936×10^{-12}	1.4211×10^{-14}	1.5100×10^{-11}	4.5581×10^{-15}
0.7	2.3511×10^{-12}	1.9096×10^{-14}	1.8000×10^{-11}	5.2496×10^{-15}
0.8	3.1481×10^{-12}	2.3537×10^{-14}	2.1100×10^{-11}	6.9760×10^{-15}
0.9	4.0962×10^{-12}	2.9310×10^{-14}	2.4200×10^{-11}	7.6462×10^{-15}
1	5.2121×10^{-12}	3.5971×10^{-14}	2.7600×10^{-11}	9.1104×10^{-15}

The results in Table 7 demonstrate the superiority of the one-step methods developed in this study, with m values of 1, 2, and 3, over the two-step method derived with three off-step points in [16]. This is noteworthy, especially considering that the method with m =1 achieves an order p of 5, while the m =2 method attains the same order p of 6.

Table 8: Absolute error y_1 for problem 6 using $h = 0.1$

x	**Error y_1 in NMm1**	**Error y_1 in NMm2**	**Error y_1 in NMm3**
0.1	1.8408×10^{-13}	8.6597×10^{-15}	2.6645×10^{-15}
0.2	9.4969×10^{-13}	3.5971×10^{-14}	3.0198×10^{-14}
0.3	2.6710×10^{-12}	7.7716×10^{-14}	7.4385×10^{-14}
0.4	5.7079×10^{-12}	1.3323×10^{-13}	1.3900×10^{-13}
0.5	1.0403×10^{-11}	2.0473×10^{-13}	2.2205×10^{-13}
0.6	1.7055×10^{-11}	2.9221×10^{-13}	3.2552×10^{-13}
0.7	2.5924×10^{-11}	3.9280×10^{-13}	4.4542×10^{-13}
0.8	3.7225×10^{-11}	4.9982×10^{-13}	5.7310×10^{-13}
0.9	5.1122×10^{-11}	6.1284×10^{-13}	7.0921×10^{-13}
1	6.7726×10^{-11}	7.3253×10^{-13}	8.5443×10^{-13}

Table 9: Absolute error y_2 for problem 6 using $h = 0.1$

x	Error y_2 in NMm1	Error y_2 in NMm2	Error y_2 in NMm3
0.1	9.1704×10^{-14}	4.7740×10^{-15}	2.3315×10^{-15}
0.2	4.7618×10^{-13}	1.9429×10^{-14}	1.7764×10^{-14}
0.3	1.3372×10^{-12}	4.0301×10^{-14}	3.8747×10^{-14}
0.4	2.8543×10^{-12}	6.7280×10^{-14}	6.9944×10^{-14}
0.5	5.2013×10^{-12}	1.0247×10^{-13}	1.1124×10^{-13}
0.6	8.5281×10^{-12}	1.4666×10^{-13}	1.6420×10^{-13}
0.7	1.2963×10^{-11}	1.9806×10^{-13}	2.2504×10^{-13}
0.8	1.8613×10^{-11}	2.5136×10^{-13}	2.8788×10^{-13}
0.9	2.5561×10^{-11}	3.0653×10^{-13}	3.5416×10^{-13}
1	3.3863×10^{-11}	3.6571×10^{-13}	4.2688×10^{-13}

The absolute errors, as displayed in Tables 8 and 9, result from solving problem 6 using the recently developed methods with m values of 1, 2, and 3, all with a step length of h = 0.1. These tables effectively demonstrate the accuracy of the new methods, which is attributable to the meticulous selection of optimal points during the derivation process.

6 Conclusion

This paper presents a novel family of implicit one-step optimized third derivative hybrid block methods, designed for the direct solution of general second-order initial value problems (IVPs). The main goal of this study was to assess the accuracy and effectiveness of these methods in solving linear and nonlinear IVPs. Thus, because of the zero-stability and A-stability of these methods, it is suitable for solving stiff IVPs as well as non-stiff IVPs. Numerical simulations were conducted on both system and non-system stiff IVPs. It is evident from the study as indicated in Table 2 to Table 7 that the new methods give accurate results in a computationally efficient manner that performs better than the existing ones based on the results produced. The incorporation of optimization techniques has notably elevated both accuracy and stability in addressing differential equations. Therefore, it is concluded that these newly derived methods are computationally reliable in solving general second-order problems of the form in equation (1). The apparent success of these methods can be attributed to the use of the optimal points. This work contributes to the existing body of literature on optimization tools for solving linear and non-linear problems of IVPs. For further research, these methods shall be applied to the problems of electric circuits to investigate the efficiency and accuracy of the proposed methods which is the main requirement of such types of problems.

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Conflict of Interest

The author declares that there is no conflict of interest regarding the publication of this paper.

References

[1] Abdelrahim R. and Omar Z. Direct solution of second-order ordinary differential equation using a single-step hybrid block method of order five. Mathematical and Computational Applications, 21(2), 12, 2016.

[2] Abderahim R., Omar Z., Ala'yed O. and Batiha B. Hybrid third derivative block method for the solution of general second order initial value problems with generalized one step point. European Journal of Pure and Applied Mathematics, 12(3), 2019; pp 1199–1214.

[3] Shokri A. The symmetric p-stable hybrid obrenchkoff methods for the numerical solution of second order ivps. TWMS J. Pure Appl. Math, 5(1), 2012.

[4] Dahlquist G. G. A special stability problem for linear multistep methods. BIT Numerical Mathematics, 3(1), 1963; pp 27–43.

[5] Lambert J. D. Numerical methods for ordinary differential systems. 146. Wiley New York, 1991.

[6] Lawal K. O, Yahaya Y. A. and Yakubu S. D. Four-step block method for solving third order ordinary differential equation. International Journal of Mathematics Trends and Technology, 57(5), 2018; pp 331–344.

[7] Li Q. and Wu X. A two-step explicit p-stable method for solving second-order initial value problems. Applied mathematics and computation, 138(2-3), 2003; pp 435–442.

[8] Singh G. and Ramos H. An optimized two-step hybrid block method formulated in variable step size mode for integrating y^"=f(x,y,y^') numerically. Numer. Math. Theor. Meth. Appl, 12(2), 2019; pp 640–660.

[9] Olabode B. T. and Momoh A. L. Continuous hybrid multistep methods with Legendre basis function for direct treatment of second order stiff odes. American Journal of Computational and Applied Mathematics, 6(2), 2016; pp 38–49.

[10] Olabode B. T. and Momoh A. L. Chebyshev hybrid multistep method for directly solving second-order initial and boundary value problems. Journal of the Nigerian Mathematical Society, 39(1), 2020; pp 97–115.

[11] Singla R., Singh G., Kanwar V. and Ramos H. Efficient adaptive step-size formulation of an optimized two-step hybrid block method for

directly solving general second-order initial-value problems. Computational and Applied Mathematics, 40(6):220, 2021.

[12] Orakwelu M. G., Goqo S. and Motsa S. An optimized two-step block hybrid method with symmetric intra-step points for second-order initial value problems. Engineering Letters, 29(3), 2021; pp 948–956.

[13] Ramos H., Kalogiratou Z, Monovasilis Th. and Simos T. E. An optimized two-step hybrid block method for solving general second order initial-value problems. Numerical Algorithms, 72, 2016; pp 1089–1102.

[14] Ramos H. and Singh G. Solving second-order two-point boundary value problems accurately by a third derivative hybrid block integrator. Applied Mathematics and Computation, 421:126960, 2022.

[15] Rufai M. A. and Ramos H. One-step hybrid block method containing third derivatives and improving strategies for solving Bratu's and Troesch's problems. Numerical Mathematics: Theory, Methods & Applications, 13(4), 2020.

[16] Awari Y. S. Some generalized two-step block hybrid numerov method for solving general second order ordinary differential equations without predictors. Science World Journal, 12(4), 2017; pp 12–18.

[17] Obarhua F. O. An efficient two-step symmetric hybrid block method for solving second order initial value problems of ordinary differential equations. Int. J. of Res. and Sci. Innovation, 6:200, 2019.

[18] Omole E. O. and Ogunware B. G. 3-point single hybrid block method (3pshbm) for direct solution of general second order initial value problem of ordinary differential equations. Journal of Scientific Research and Reports, 20(3), 2018; pp 1–11.

[19] Skwame Y., Sabo J. and Mathew M. The treatment of second-order ordinary differential equations using equidistant one-step block hybrid. Asian Journal of Probability and Statistics, 5(3), 2019; pp 1–9.

[20] Yakubu S. D., Yahaya Y. A. and Lawal K. O. 3-step block hybrid linear multistep methods for solution of special second order ordinary differential equations. Journal of the Nigerian Mathematical Society, 40(2), 2021; pp 149–160.

عائلة من طرائق التحسين المستقرة المهجنة من المشتقة الثالثة ذات الخطوة الواحدة لحل المسائل القيم الابتدائية العامة من الرتبة الثانية

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الملخص

نقدم في هذا البحث ثلاثة عناصر من عائلة طرائق التحسين الأمثل للكتل الهجينة المستقرة ذات الخطوة الواحدة لغرض حل مسائل القيم الابتدائية العامة من الدرجة الثانية، حيث يتم دمج منهجية التحسين في عملية الاشتقاق للحصول على الدقة بشكل افضل وذلك من خلال التحليل الدقيق، كما أثبت ان جميع الطرائق المشتقة كانت ذات استقرار صفري وثابتة (متسقة) ومستقرة ومتقاربة. كما وقد تم التحقق من صحة تنفيذ هذه الطرائق المشتقة الحديثة من خلال التجارب العددية وقد أظهرت النتائج دقة فائقة مقارنة ببعض الطرائق العددية الحالية والتي تم الكشف عنها في الدراسة، وقد تبين ان كل طرق الكتلة الهجينة المحسنة من المشتقة الثالثة والتي تم الحصول عليها حديثاً انها تمتلك ثوابت خطأ صغيرة جداً وذات دقة عالية.

الكلمات المفتاحية: طريقة الاشتقاق الثالث الهجين الأمثل.