Hosoya Polynomial and Wiener Index of Zero-Divisor Graph of Z_n

Husam Q. Mohammad **Mohammad N. Authman**
husam_alsabawi@yahoo.com
College of Computer Sciences and Mathematics
University of Mosul, Mosul, Iraq

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ABSTRACT

Let R be a commutative ring with identity. We associate a graph Γ(R). In this paper, we find Hosoya polynomial and Wiener index of $\Gamma(Z_n)$, with $n= p^m$ or $n= p^mq$, where p and q are distinct prime numbers and m is an integer with m≥2.

Keywords: *Zero-divisor graph, commutative rings, Hosoya polynomial and Wiener index.*

متعددة حدود هوسويا ودليل وينر لبيات قاسم الصفر لحلقات Z_n

حسام قاسم محمد ***م. م. محمد نوفل عثمان***

كلية علوم الحاسبات والرياضيات

جامعة الموصل، الموصل، العراق

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الملخص

لتكن R حلقة ابدالية بعنصر محايد. نمثل البيان Γ(R). في هذا البحث وجدنا متعددة حدود هوسويا ودليل وينر للبيان $\Gamma(Z_n)$ حيث $n=p^mq$ أو $n= p^m$, بحيث أن p و q عددان أوليان مختلفان وان m عدد صحيح موجب أكبر او يساوي 2 .

الكلمات المفتاحية : بيان قاسم الصفر، الحلقات الابدالية، متعددة حدود هوسويا ودليل وينر.

1. Introduction

Let R be a commutative ring with identity, and let Z(R) be the set of all zero-divisors in R, and $Z^*(R)$ is the set of all non-zero zero-divisors in it. We associate a simple graph Γ(R) to R with vertices $Z^*(R)$, and for two distinct vertices $x,y\epsilon Z^*(R)$, there is an edge connecting x and y if and only if xy= 0.

The notion of a zero divisor graph of a commutative ring was first introduced in 1988 by Beck in [5], where he was interesting in colorings. This investigation of coloring of a commutative ring was then continued by Anderson and Naseer in [3], and further Anderson and Livingston in [2] associate a graph Γ(R) to R. The principal ideal of an R is an ideal that is generated by one element of R, say a, and usually denoted by (a). The ring R is called local ring if it contains exactly one maximal ideal.

A graph G is said to be connected [6] if there is a path between any two distinct vertices of G. For vertices x and y of G, let d(x, y) be the length of a shortest path from x to y. The diameter of G is defined by diam(G)= max{d(x,y) : x,y ∈ V(G)}, where V(G) is the set of all vertices of G. A graph is complete if every two of its vertices are adjacent, so the complete graph of order n is denoted by K_n. The complement $\overline{K}_n$ of the complete graph K_n has n vertices and no edges, and is referred to as the empty graph of order n. The subsets $V_1, V_2, \dots, V_r$, $r \geq 2$, are called r-partite of the set V(G), if V_i is non empty, and the intersection between V_i and V_j is empty for any $1 \leq i, j \leq r$ with $i \neq j$, so that $\bigcup_{i=1}^{r} V_i = V(G)$.

Hosoya polynomial of the graph G is defined by : $H(G ; x) = \sum_{k=0}^{diam(G)} d(G,k)x^k$, where $d(G,k)$ is the number of pairs of vertices of a graph G that are at distance k apart, for k=0,1,2, . . ., diam(G). The Wiener index of G is defined as the sum of all distances between vertices of the graph, and denoted by W(G), and we can find this index by differentiating Hosoya polynomial for the given distance with respect to x and putting x = 1. See [7, 9].

As usual we shall assume that p and q are distinct positive prime numbers and m be an integer with m≥2. In [1] Ahmadi and Nezhad proved some results concerning the Wiener index of $\Gamma(Z_n)$, where $n = p^2$, pq and p^2q. In this paper we extended these results to $n= p^m$, p^mq .

2. Hosoya Polynomial and Wiener Index of $\Gamma(Z_{p^m})$

In this section, we find the Hosoya polynomial and the Wiener index of $\Gamma(Z_{p^m})$. It is clear that $Z^*(Z_{p^m})$= (p)\{ 0 }={ p, 2p, 3p, . . ., $(p^{m-1} - 1)$ p}, so we have $|Z^*(Z_{p^m})| = p^{m-1} - 1$. We shall begin this section with the following lemma :

Lemma 2.1 [8, Lemma 2.1.] **:** Let Z_n be a ring of integers modulo n. Then, the number of all non-zero zero-divisors for k|n are $\frac{n}{k} - 1$.

Theorem 2.2 : $\Gamma(Z_{p^3}) \cong K_{p-1} + \overline{K}_{p^2-p}$.

Proof : Since p is a prime number, then it is clear that the ring Z_{p^3} is a local ring, so we have $Z^*(Z_{p^3})$ = (p)\{ 0 } = { kp : k = 1,2,3, . . . , $p^2 - 1$ }.

Now, we can classify $Z^*(Z_{p^3})$ into the two disjoint subsets as follows :

A_1= (p^2) \{ 0 }, and A_2= (p)\ { $A_1 \cup$ { 0 }}. It is clear that $Z^*(Z_{p^3}) = A_1 \cup A_2$ and by using Lemma 2.1 we have $|A_1| = \frac{p^3}{p^2} - 1 = p - 1$, and $|A_2| = \frac{p^3}{p} - (\frac{p^3}{p^2} - 1 + 1) = p^2 - p$, so we can write $A_1=\{k_1p^2: k_1=1,2,\dots,p-1\}$ and $A_2=\{k_2p: k_2=1,2,\dots,p^2-1 ; p \nmid k_2\}$.

Now, let $x,y \in Z^*(Z_{p^3})$. Then, there are three cases :

Case 1: If $x,y \in A_1$, then there exists positive integers k_1 and k_2 with $p \nmid k_1,k_2$ such that $x= k_1 p^2$ and $y= k_2 p^2$, and we have

$xy= k_1 p^2 k_2 p^2 = k_1 k_2 p^4 \equiv 0 \pmod{p^3}$, then x adjacent with y in this case .

Case 2: If $x \in A_1$ and $y \in A_2$, then there exists positive integers k_1 and k_2 with $p \nmid k_1,k_2$ such that $x = k_1 p^2$, and $y = k_2 p$, and we have

$xy = k_1 p^2 k_2 p = k_1 k_2 p^3 \equiv 0 \pmod{p^3}$, then x adjacent with y in this case .

Case 3: If $x,y \in A_2$, then there exists positive integers k_1 and k_2 with $p \nmid k_1,k_2$ such that $x= k_1 p$ and $y= k_2 p$, and we have $xy = k_1 p k_2 p = k_1 k_2 p^2 \not\equiv 0 \pmod{p^3}$, then x and y are not adjacent in this case.

From the previous, we see that every vertex in A_1 is adjacent with any other vertex in A_1 and A_2, so that no vertex in A_2 is adjacent with any other vertex in A_2, therefore we have : $\Gamma(Z_{p^3}) \cong K_{|A_1|} + \overline{K}_{|A_2|} = K_{p-1} + \overline{K}_{p^2-p}$. ■

Theorem 2.3: $H(\Gamma(Z_{p^3}); x) = a_0 + a_1 x + a_2 x^2$, where $a_0 = p^2 - 1$, $a_1 = \frac{1}{2}(2p^3 - 3p^2 - p + 2)$, and $a_2 = \frac{1}{2}(p^4 - 2p^3 + p)$.

Proof : From clearly that $diam(\Gamma(Z_{p^3})) = d(x,y) = 2$, for all $x,y \in A_2$, therefore $H(\Gamma(Z_{p^3}), x) = a_0 + a_1 x + a_2 x^2$, where $a_i = d(\Gamma(Z_{p^3}), i)$ for $i = 0,1,2$. It is clear that $a_0 = d(\Gamma(Z_{p^3}), 0) = |Z^*(Z_{p^3})| = p^2 - 1$.

Now, let $Z^*(Z_{p^3}) = A_1 \cup A_2$, where $A_1 = (p^2)\backslash\{0\}$ and $A_2 = (p)\backslash\{A_1 \cup \{0\}\}$ and by Lemma 2.1 we have, $|A_1| = p - 1$, and $|A_2| = p^2 - p$.

To find a_1, let $x,y \in Z^*(Z_{p^3})$ such that $d(x,y) = 1$, from the proof of Theorem 2.2 we get that $d(x,y) = 1$ if and only if $x,y \in A_1$ or $x \in A_1$ and $y \in A_2$, then we have :

$$a_1 = d(\Gamma(Z_{p^3}), 1) = \binom{|A_1|}{2} + |A_1||A_2| = \binom{p-1}{2} + (p-1)(p^2 - p) = \frac{1}{2}(2p^3 - 3p^2 - p + 2).$$

To find a_2, let $x,y \in Z^*(Z_{p^3})$ such that $d(x,y) = 2$, from the proof of Theorem 2.2, we have $d(x,y) = 2$ if and only if $x,y \in A_2$, then we have :

$$a_2 = d(\Gamma(Z_{p^3}), 2) = \binom{|A_2|}{2} = \binom{p^2 - p}{2} = \frac{1}{2}(p^4 - 2p^3 + p). \blacksquare$$

Corollary 2.4 : $W(\Gamma(Z_{p^3})) = \frac{1}{2}(2p^4 - 2p^3 - 3p^2 + p + 2)$.

Proof : Since $W(\Gamma(Z_{p^3})) = \frac{d}{dx} H(\Gamma(Z_{p^3}); x)|_{x=1}$, then we have $W(\Gamma(Z_{p^3})) = 0 + \frac{1}{2}(2p^3 - 3p^2 - p + 2) + 2x(\frac{1}{2}(p^4 - 2p^3 + p))|_{x=1}$

$$= \frac{1}{2}(2p^4 - 2p^3 - 3p^2 + p + 2). \blacksquare$$

Next, we give the following definition .

Definition 2.5 : Let Z_{p^m} be the ring of integers modulo p^m. Then we can write $Z^*(Z_{p^m}) = \bigcup_{i=1}^{m-1} A_i$, where A_i are disjoint subsets of $Z^*(Z_{p^m})$, for $1 \le i \le m-1$, which are defined as follows :

$A_1 = (p^{m-1})\backslash\{0\}$, $A_2 = (p^{m-2})\backslash\{A_1 \cup \{0\}\}$, $A_3 = (p^{m-3})\backslash\{A_1 \cup A_2 \cup \{0\}\}, \ldots,$
$A_{m-1} = (p)\backslash\{\{\bigcup_{i=1}^{m-2} A_i\} \cup \{0\}\}$.

Notice that, from Lemma 2.1, we get
$|A_i| = p^i - p^{i-1}$, for any $1 \le i \le m-1$, so that we can write
$A_i = \{k_i p^{m-i} : k_i = 1,2,\ldots,p^i - 1 ; p \nmid k_i\}$, for any $1 \le i \le m-1$.

Lemma 2.6 : Let A_i, for $1 \le i \le m-1$ be subsets of $Z^*(Z_{p^m})$ which are defined in Definition 2.5 and let s and t are two integers with $1 \le s \le t \le m-1$, then $\sum_{i=s}^{t}|A_i| = p^t - p^{s-1}$.

Proof : Since, $|A_i| = p^i - p^{i-1}$, $\forall\, 1 \le i \le m-1$, then we have

$$\sum_{i=s}^{t}|A_i| = \sum_{i=s}^{t}(p^i - p^{i-1}) = p^s - p^{s-1} + p^{s+1} - p^s + \ldots + p^{t-1} - p^{t-2} + p^t - p^{t-1}$$
$$= p^t - p^{s-1}. \blacksquare$$

Theorem 2.7 : Let A_i, for $1 \le i \le m-1$, be subsets of $Z^*(Z_{p^m})$ which are defined in Definition 2.5. Then, for any $x,y \in Z^*(Z_{p^m})$, $xy = 0$ if and only if $x \in A_i$ and $y \in A_j$ such that $i + j \le m$, for some $1 \le i, j \le m-1$.

Proof : From Definition 2.5 we have $Z^*(Z_{p^m}) = \bigcup_{i=1}^{m-1} A_i$, where $A_i = \{k_i p^{m-i} : k_i = 1,2,\ldots,p^i - 1 ; p \nmid k_i\}$, for $1 \le i \le m-1$. Now, for any $1 \le i, j \le m-1$, let $x \in A_i$ and $y \in A_j$. Then, there exists two positive integers k_i and k_j such that $x = k_i p^{m-i}$ and $y = k_j p^{m-j}$, with $p \nmid k_i, k_j$.

Now, if $xy=0$. Then, $xy = k_i p^{m-i} k_j p^{m-j} = k_i k_j p^{2m-(i+j)} \equiv 0 \pmod{p^m}$, and since $k_i k_j \not\equiv 0 \pmod{p^m}$, therefore $p^{2m-(i+j)} \equiv 0 \pmod{p^m}$, and that means p^m divides $p^{2m-(i+j)}$, which implies that $2m-(i+j) \ge m$, therefore $i + j \le m$.

Conversely: Let $x\in A_i$ and $y\in A_j$ such that $i + j\leq m$ for some $1\leq i , j \leq m-1$, and suppose contrary that $xy\neq 0 \Rightarrow xy= k_i k_j p^{2m-(i+j)} \not\equiv 0 \pmod{p^m}$, and since, $p \nmid k_i,k_j$, therefore $p^m \nmid p^{2m-(i+j)}$. Then, we get $2m-(i+j) < m$, so that $2m - m < i+j$, which implies that $i+j > m$, this contradiction, therefore $xy=0$. ■

From Theorem 2.7 and Lemma 2.6 we can give the general form of the graph $\Gamma(Z_{p^t})$, where t=4,5 **,** as the following **:**

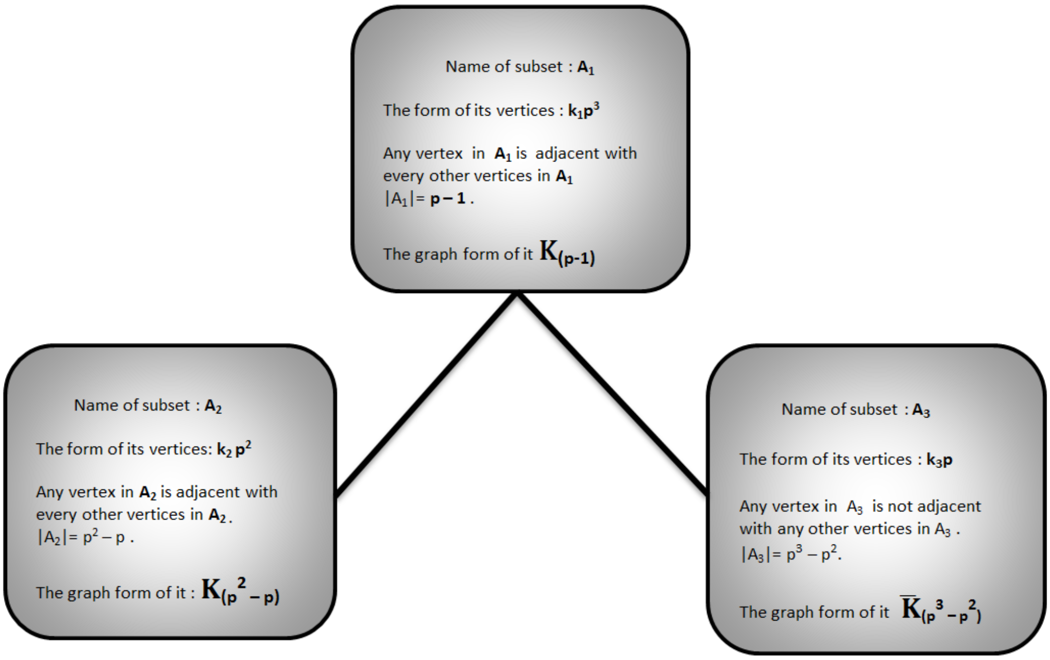


Figure 2.1 : The general form of the graph $\Gamma(Z_{p^4}) \cong K_{(p-1)} + (K_{(p^2-p)} \cup \overline{K}_{(p^3-p^2)})$

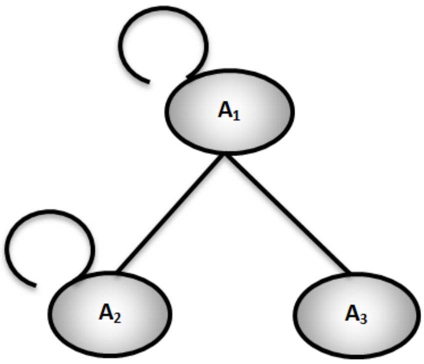


Figure 2.2
The general form of the graph $\Gamma(Z_{p^4})$

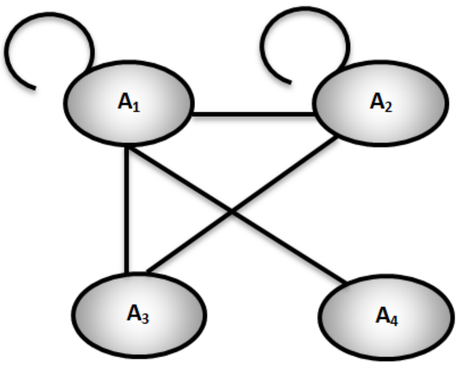


Figure 2.3
The general form of the graph $\Gamma(Z_{p^5})$

We can now give the general form of the graph $\Gamma(Z_{p^m})$:

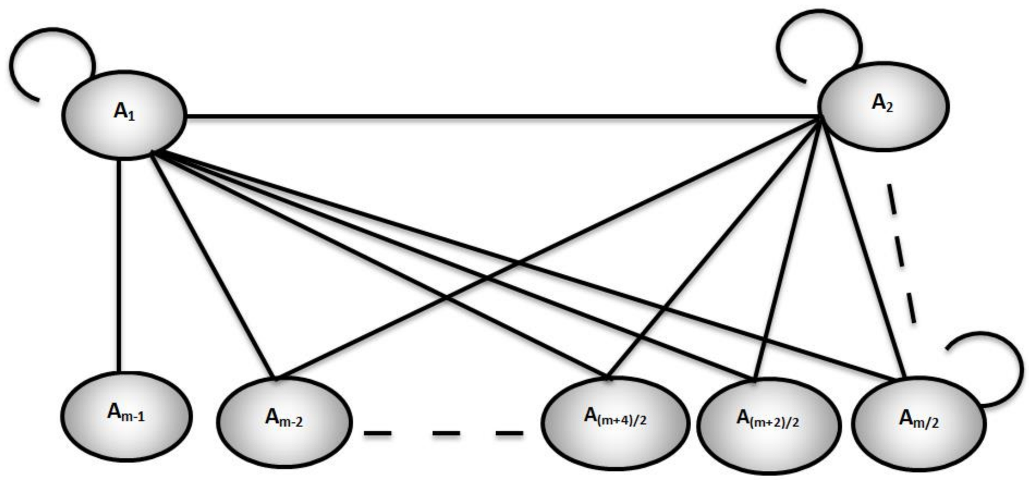


Figure 2.4: The general form of the graph $\Gamma(Z_{p^m})$, where m is an even number with m≥6 .

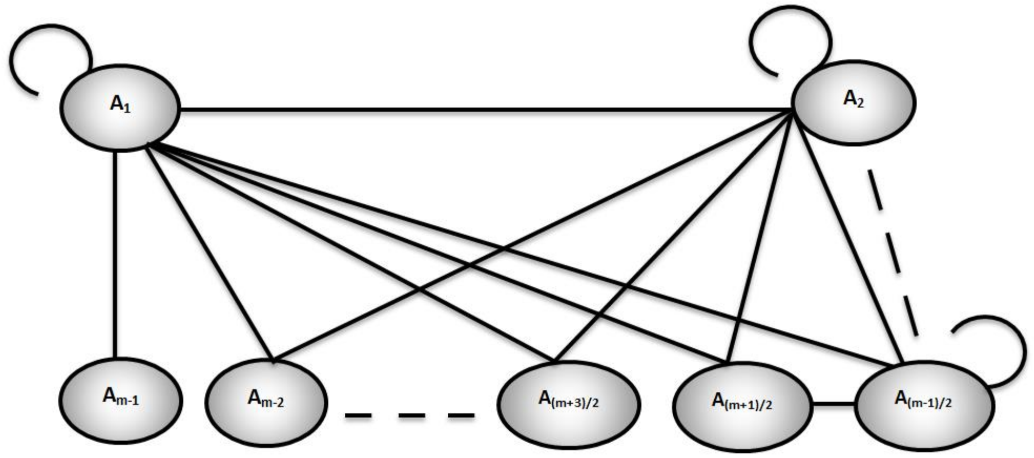


Figure 2.5 : The general form of the graph $\Gamma(Z_{p^m})$, where m is an odd number with m≥7 .

<u>Theorem 2.8</u> : The graph $\Gamma(Z_{p^m})$ is s-partite graph, where

$$s = \begin{cases} p^{\frac{m-1}{2}} & ;\ \text{if m is an odd number} \\ p^{\frac{m}{2}} - 1 & ;\ \text{if m is an even number} \end{cases}.$$

Proof : From Definition 2.5, we have $Z^*(Z_{p^m})= \bigcup_{i=1}^{m-1} A_i$, where $A_i=\{k_i\, p^{m-i}, k_i=1,2,\ldots, p^i-1\ ;\ p\nmid k_i\}$, for $1\le i \le m-1$.

Suppose that m is an odd number, we see that by Theorem 2.7, any two distinct vertices lie in $\bigcup_{i=1}^{\frac{m-1}{2}} A_i$ are adjacent because that $i + j\le m$, for any $1\le i, j \le \frac{m-1}{2}$, this means that, we cannot put the vertices of the sets $A_1, A_2, \ldots, A_{\frac{m-1}{2}}$ in less than $\sum_{i=1}^{\frac{m-1}{2}} |A_i| = p^{\frac{m-1}{2}} - 1$ of partite sets. also by Theorem 2.7 we see that any vertex $x\in A_{\frac{m+1}{2}}$ is adjacent with every vertex of $\bigcup_{i=1}^{\frac{m-1}{2}} A_i$ because that $\frac{m+1}{2} + i \le m$, for any $1\le i \le \frac{m-1}{2}$, so that x is not adjacent with any other vertex in $A_{\frac{m+1}{2}}$ because that $2(\frac{m+1}{2}) > m$, therefore we must consider new partite set, say V, contains the vertices of $A_{\frac{m+1}{2}}$, in this case, we cannot put the vertices of the sets $A_1, A_2, \ldots, A_{\frac{m+1}{2}}$, in less than $(p^{\frac{m-1}{2}} -$

1)+1= $p^{\frac{m-1}{2}}$ of partite sets. Now, if we can put the vertices of $\bigcup_{i=\frac{m+3}{2}}^{m-1} A_i$ in V, then the theorem hold, that is : by Theorem 2.7 we see that any two distinct vertices in $\bigcup_{i=\frac{m+3}{2}}^{m-1} A_i$ are not adjacent because that i+j > m for any $\frac{m+3}{2} \leq i, j \leq m-1$, so that any vertex in V is not adjacent with every vertex of $\bigcup_{i=\frac{m+3}{2}}^{m-1} A_i$ because that $\frac{m+1}{2}$ + i > m, for any $\frac{m+3}{2} \leq i \leq m-1$, and this shows that we cannot put the vertices of $Z^*(Z_{p^m}) = \bigcup_{i=1}^{m-1} A_i$ in less than $p^{\frac{m-1}{2}}$ of partite sets, therefore $\Gamma(Z_{p^m})$ is $p^{\frac{m-1}{2}}$-partite graph.

Now, let m be an even integer number, similarly we cannot put the vertices of the set $\bigcup_{i=1}^{\frac{m}{2}} A_i$ in less than $\sum_{i=1}^{\frac{m}{2}} |A_i| = p^{\frac{m}{2}} - 1$ of partite sets, say $V_1, V_2, \ldots, V_{p^{\frac{m}{2}}-1}$, each of these partite sets contains only one vertex of the set $\bigcup_{i=1}^{\frac{m}{2}} A_i$, suppose that the partite set $V_{p^{\frac{m}{2}}-1}$ contains one of the vertices of the set $A_{\frac{m}{2}}$, and we are going to show that we can put the vertices of the set $\bigcup_{i=\frac{m+2}{2}}^{m-1} A_i$ in the partite set $V_{p^{\frac{m}{2}}-1}$, that is : by Theorem 2.7 we see that any two distinct vertices in the set $\bigcup_{i=\frac{m+2}{2}}^{m-1} A_i$ are not adjacent because that i+j>m for any $\frac{m+2}{2} \leq i, j \leq m-1$, so that any vertex of the set $\bigcup_{i=\frac{m+2}{2}}^{m-1} A_i$ is not adjacent with every vertex of the set $A_{\frac{m}{2}}$ because that $\frac{m}{2}$ + i > n for any $\frac{m+2}{2} \leq i \leq m-1$, and this shows we can put the vertices of the set $\bigcup_{i=\frac{m+2}{2}}^{m-1} A_i$ in the partite set $V_{p^{\frac{m}{2}}-1}$, therefore we cannot put the vertices of $Z^*(Z_{p^m}) = \bigcup_{i=1}^{m-1} A_i$ in less than $p^{\frac{m}{2}} - 1$ of partite sets, hence $\Gamma(Z_{p^m})$ is $(p^{\frac{m}{2}} - 1)$-partite graph. ■

Lemma 2.9 [7] : Let G be a connected graph of order r. Then $\sum_{i=0}^{diam(G)} d(G, i) = \frac{1}{2} r(r+1)$.

Now, we give the main result in this section.

Theorem 2.10: $H(\Gamma(Z_{p^m}); x) = a_0 + a_1 x + a_2 x^2$, where

$a_0 = p^{m-1} - 1$,

$a_1 = \frac{1}{2}[(m-1)p^m - m p^{m-1} - p^{\lfloor \frac{m}{2} \rfloor} + 2]$, and

$a_2 = \frac{1}{2}[\, p^{2(m-1)} - (m-1)p^m + (m-3)p^{m-1} + p^{\lfloor \frac{m}{2} \rfloor}\,]$.

Proof : When m= 2, we have $\Gamma(Z_{p^2}) \cong K_{p-1}$, and the theorem is true in this case.

Now, suppose that m≥3, since Z_{p^m} is a local ring, then by [4, Theorem 2.3.], there is a vertex adjacent with every other vertices in $\Gamma(Z_{p^m})$, this means that $diam(\Gamma(Z_{p^m})) = 2$, therefore $H(\Gamma(Z_{p^m}); x) = a_0 + a_1 x + a_2 x^2$, where $a_i = d(\Gamma(Z_{p^m}), i)$, for i = 0,1,2 .

To find a_0, by Lemma 2.1 we have

$a_0 = d(\Gamma(Z_{p^m}), 0) = |Z^*(Z_{p^m})| = \frac{p^m}{p} - 1 = p^{m-1} - 1$.

To find a_1, suppose that m be an odd number, and let $x, y \in Z^*(Z_{p^m})$, since $Z^*(Z_{p^m}) = \bigcup_{i=1}^{m-1} A_i$, then by Theorem 2.7 we see that d(x,y)=1 (i.e. xy=0) if and only if

$x \in A_i$ and $y \in A_j$ such that $i + j \leq m$, for some $1 \leq i, j \leq m-1$, and this holds if and only if one of the following two cases holds :

Case 1 : $1 \leq i, j \leq \frac{m-1}{2}$, because that $i+j \leq m$ for any $1 \leq i,j \leq \frac{m-1}{2}$, in this case there are m_1 edges where

$$m_1 = \binom{\sum_{i=1}^{\frac{m-1}{2}} |A_i|}{2} = \binom{p^{\frac{m-1}{2}}-1}{2} = \frac{1}{2}(p^{\frac{m-1}{2}} - 1)(p^{\frac{m-1}{2}} - 2) \quad \ldots (*) .$$

Case 2 : $1 \leq i \leq \frac{m-1}{2}$ and $\frac{m+1}{2} \leq j \leq m - i$, since that $i+j \leq m$ for any $1 \leq i \leq \frac{m-1}{2}$ and $\frac{m+1}{2} \leq j \leq m - i$, in this case there are m_2 edges where

$m_2 = \sum_{i=1}^{\frac{m-1}{2}} (|A_i| \sum_{j=\frac{m+1}{2}}^{m-i} |A_j|)$, since $|A_i| = p^i - p^{i-1}$, for each $1 \leq i \leq m-1$, and by using Lemma 2.6 we get :

$$m_2 = \sum_{i=1}^{\frac{m-1}{2}} (p^i - p^{i-1})(p^{m-i} - p^{\frac{m-1}{2}})$$

$$= \sum_{i=1}^{\frac{m-1}{2}} p^{i-1}(p-1)(p^{m-i} - p^{\frac{m-1}{2}}) = \sum_{i=1}^{\frac{m-1}{2}} (p-1)(p^{m-1} - p^{\frac{m-3}{2}} p^i)$$

$$= \sum_{i=1}^{\frac{m-1}{2}} p^{m-1}(p-1) - p^{\frac{m-3}{2}}(p-1) \sum_{i=1}^{\frac{m-1}{2}} p^i$$

$= \frac{m-1}{2} p^{m-1}(p-1) - p^{\frac{m-3}{2}}(p-1) \sum_{i=1}^{\frac{m-1}{2}} p^i$, and since $\{p^i\}_{i=1}^{\frac{m-1}{2}}$ is a geometric sequence, therefore we can use $\sum_{i=1}^{k} a^i = \frac{a^{k+1}-a}{a-1}$ where a be any real number and k is any positive integer, hence we have : $m_2 = \frac{m-1}{2} p^{m-1}(p-1) - p^{\frac{m-3}{2}}(p-1) \frac{p^{\frac{m+1}{2}}-p}{(p-1)}$

$$= \frac{m-1}{2} p^{m-1}(p-1) - p^{\frac{m-1}{2}}(p^{\frac{m-1}{2}} - 1) \quad \ldots (**).$$

Now, from (*) and (**), we get

$$a_1 = m_1 + m_2 = \frac{1}{2}(p^{\frac{m-1}{2}} - 1)(p^{\frac{m-1}{2}} - 2) + \frac{m-1}{2} p^{m-1}(p-1) - p^{\frac{m-1}{2}}(p^{\frac{m-1}{2}} - 1)$$

$$= \frac{1}{2}[(m-1)p^m - m p^{m-1} - p^{\frac{m-1}{2}} + 2].$$

Similarly, when an m be an even number we get that $a_1 = \frac{1}{2}[(m-1)p^m - m p^{m-1} - p^{\frac{m}{2}} + 2]$.

Hence $a_1 = \frac{1}{2}[(m-1)p^m - m p^{m-1} - p^{\lfloor \frac{m}{2} \rfloor} + 2]$.

Next, to find a_2 we shall use lemma 2.9, and we get :

$$a_2 = \frac{1}{2} a_0 (a_0+1) - a_0 - a_1$$

$$= \frac{1}{2}(p^{m-1} - 1) p^{m-1} - (p^{m-1} - 1) - \frac{1}{2}[(m-1)p^m - m p^{m-1} - p^{\lfloor \frac{m}{2} \rfloor} + 2]$$

$$= \frac{1}{2}[p^{2(m-1)} - (m-1)p^m + (m-3)p^{m-1} + p^{\lfloor \frac{m}{2} \rfloor}] . \quad \blacksquare$$

Corollary 2.11: $W(\Gamma(Z_{p^m})) = \frac{1}{2}[2p^{2(m-1)} - (m-1)p^m + (m-6)p^{m-1} + p^{\lfloor \frac{m}{2} \rfloor} + 2]$.

3. Hosoya Polynomial and Wiener Index of $\Gamma(Z_{p^m q})$.

In this section, we find the Hosoya polynomial and the Wiener index of $\Gamma(Z_{p^m q})$. First, we shall give the following lemma :

Lemma 3.1 : The number of all non-zero zero-divisors of a ring $Z_{p^m q}$ is $(p+q-1)\, p^{m-1} - 1$.

Proof : Since, p and q are distinct prime numbers, then clearly $Z(R)= (p)\cup(q)$, therefore $Z^*(R)= \{(p)\cup(q)\} \setminus\{0\}$.

Now, let $x\in Z^*(R)$, then either $x\in(p)$ or $x\in(q)$ with $x\notin(pq)$, so by Lemma 2.1 we get :

$$|Z^*(R)|= (\tfrac{p^m q}{p} - 1) + (\tfrac{p^m q}{q} - 1) - (\tfrac{p^m q}{pq} - 1)$$
$$= (p^{m-1}q - 1) + (p^m - 1) - (p^{m-1} - 1)$$
$$= p^{m-1}q -1+ p^m - 1 - p^{m-1}+1$$
$$= (p+q-1)\, p^{m-1} -1 . \blacksquare$$

Definition 3.2 : Let $Z_{p^m q}$ be the ring of integers modulo $p^m q$, then we can write : $Z^*(Z_{p^m})= \bigcup_{i=1}^{m}(B_i \cup C_i)$, where B_i and C_i, are disjoint subsets of $Z^*(Z_{p^m q})$,for $1\le i \le m$, which are defined as follows :

$B_1= (p^{m-1}q)\setminus\{0\}$, $B_2= (p^{m-2}q)\setminus\{B_1 \cup\{0\}\}$,
$B_3= (p^{m-3}q)\setminus \{B_1 \cup B_2 \cup \{0\}\}, \ldots$,
$B_m = (q)\setminus\{\{\bigcup_{i=1}^{m-1} B_i\}\cup\{0\}\}$, and
$C_1=(p^m)\setminus\{0\}$, $C_2=(p^{m-1})\setminus\{B_1\cup C_1\cup\{0\}$,
$C_3=(p^{m-2})\setminus\{B_1\cup C_1\cup B_2\cup C_2\cup\{0\}\}, \ldots$,
$C_m = (p))\setminus\{\{\bigcup_{i=1}^{m-1}(B_i \cup C_i)\} \cup\{0\}\}$.

Notice that, by Lemma 2.1 we get :

$|B_i| = p^i - p^{i-1}$, for any $1 \le i \le m$, $|C_1| =(q-1)$ and $|C_i| =(p^{i-1}-p^{i-2})(q-1)$, for all $2\le i \le m$, also we can write :

$B_i =\{k_i p^{m-i}q : k_i=1,2,\ldots,p^i -1 ; p\nmid k_i\}$, and $C_i =\{k_i p^{m-i+1} : k_i=1,2,\ldots, p^{i-1}q -1 ; q\nmid k_i\}$, for any $1\le i \le m$.

Remarks :

(1) $\sum_{i=1}^{m}|B_i| = p^m - 1$.
(2) $\sum_{i=1}^{m}|C_i| = p^{m-1}(q - 1)$.
(3) $|C_i| =(q-1)\,|B_{i-1}|$, for any $2\le i \le m$.
(4) $|A_i| = |B_i|$, for any $1\le i \le m-1$, where A_i ,for all $1\le i \le m-1$, be subsets of $Z^*(Z_{p^m})$ which are defined in Definition 2.5 .

Lemma 3.3 : Let B_i and C_i , for all $1\le i \le m$, be subsets of $Z^*(Z_{p^m q})$ which are defined in Definition 3.2 then :

1- If s and t are two integers with $1\le s \le t \le m$, then $\sum_{i=s}^{t}|B_i| = p^t - p^{s-1}$.
2- If t be an integer with $1\le t \le m$, then $\sum_{i=1}^{t}|C_i| = (q-1)\, p^{t-1}$.
3- If s and t are two integers with $2\le s \le t \le m$, then $\sum_{i=s}^{t}|C_i| = (q-1)(p^{t-1} - p^{s-2})$.

Proof : By the same method of a proof of Lemma 2.6 . ■

Theorem 3.4 : Let B_i and C_i , for $1 \le i \le m$, be subsets of $Z^*(Z_{p^m q})$ which are defined in Definition 3.2, and let $x,y \in Z^*(Z_{p^m q})$. Then, $xy= 0$ if and only if either $x\in B_i$ and $y\in B_j$ with $i+j \le m$, or $x\in B_i$ and $y\in C_j$ with $i+j\le m+1$, for some $1 \le i, j \le m$.

Proof : From the Definition 3.2, we have $Z^*(Z_{p^m q})= \bigcup_{i=1}^{m}(B_i \cup C_i)$. Now, let $x,y \in Z^*(Z_{p^m q})$ such that $xy= 0$, since $x,y \in \bigcup_{i=1}^{m}(B_i \cup C_i)$, then there are two cases :

Case 1 : $x\in B_i$ and $y\in B_j$ for some $1 \le i, j \le m$, in this case, there are positive integers k_i and k_j with $p\nmid k_i, k_j$, such that $x=k_i\, p^{m-i}q$ and $y=k_j\, p^{m-j}q$, for some $1 \le i, j \le m$, since $xy=0$ by hypothesis, then we get $xy=(k_i k_j)p^{2m-(i+j)}q^2 \equiv 0 \pmod{p^m q}$, since $p\nmid k_i, k_j$, therefore $p^{2m-(i+j)} q^2 \equiv 0 \pmod{p^m q}$, this means that $p^{2m-(i+j)}$ is divisible by p^m. Therefore $2m-(i+j) \geq m$, hence $i+j \leq m$.

Case 2 : $x\in B_i$, and $y\in C_j$ for some $1 \leq i, j \leq m$, in this case, there are positive integers k_i and k_j with $p\nmid k_i$ and $q\nmid k_j$, such that $x= k_i p^{m-i} q$ and $y= k_j p^{m-j+1}$, for some $1 \leq i, j \leq m$, since xy=0 by hypothesis, then $xy= (k_i k_j)p^{2m-(i+j)+1} q \equiv 0 \pmod{p^m q}$, Since $p\nmid k_i$ and $q\nmid k_j$, therefore $p^{2m-(i+j)} q \equiv 0 \pmod{p^m q}$, this means that $p^{2m-(i+j)}$ is divisible by p^m, therefore $2m-(i+j)+1 \geq m$, hence $i+j \leq m+1$.

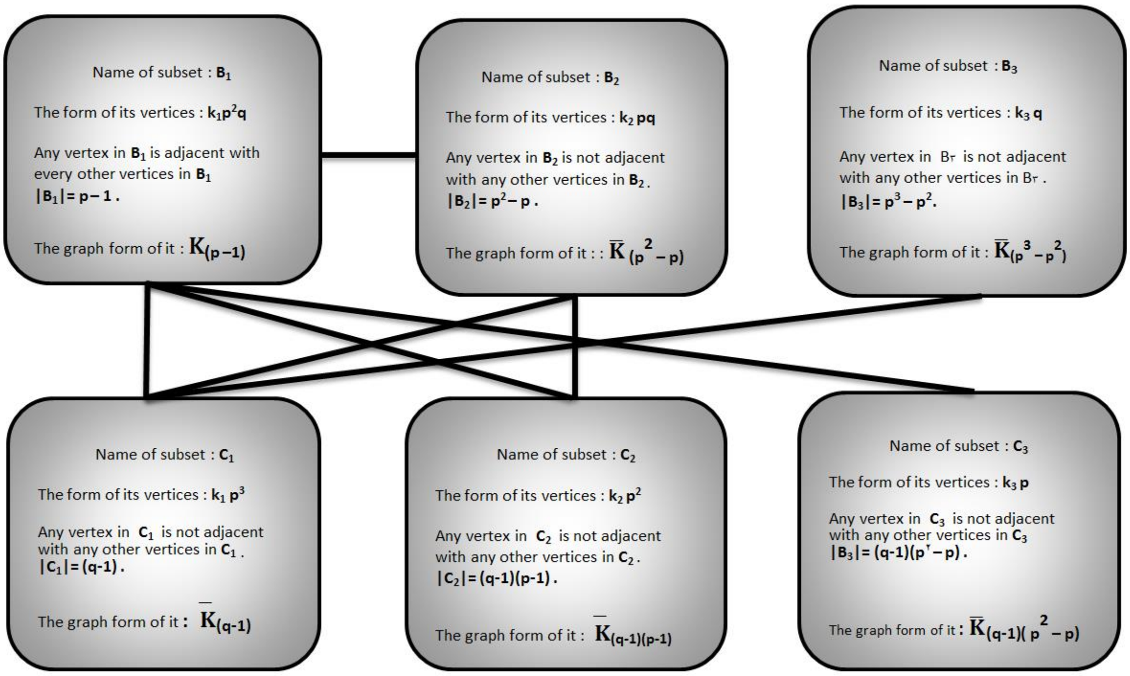
Finally, we see that when $x\in C_i$ and $y\in C_j$, then $xy\neq 0$ for any $1 \leq i, j \leq m$.

From previous, we get that if $xy= 0$, then either $x\in B_i$ and $y\in B_j$ with $i+j \leq m$, or $x\in B_i$ and $y\in C_j$ with $i+j \leq m+1$, for some $1 \leq i, j \leq m$.

Conversely : Let $x\in B_i$ and $y\in B_j$ for some $1 \leq i, j \leq m$, such that $i+j \leq m$, and suppose contrary that $xy\neq 0$,we get $xy= (k_i k_j)p^{2m-(i+j)}q^2 \not\equiv 0 \pmod{p^m q}$, since $p\nmid k_i, k_j$ and q divides q^2 then $p^{2m-(i+j)}$ is not divisible by p^m, therefore $2m-(i+j)<m \Longrightarrow i+j>m$, this contradiction, therefore must be $xy=0$.

Now, let $x\in B_i$ and $y\in C_j$ for some $1 \leq i, j \leq m$, such that $i+j \leq m+1$, and suppose contrary that $xy\neq 0$,we get $xy= (k_i k_j)p^{2m-(i+j)+1} q \not\equiv 0 \pmod{p^m q}$, and since $p\nmid k_i$ and $q\nmid k_j$ then $p^{2m-(i+j)+1}$ is not divisible by p^m, therefore $2m-(i+j)+1<m \Longrightarrow i+j>m+1$, also this is a contradiction, therefore must be $xy=0$. ■

From Theorem 3.4 and Lemma 3.3, we can give the general form of the graph $\Gamma(Z_{p^t q})$, where t=3,4 , as follows :



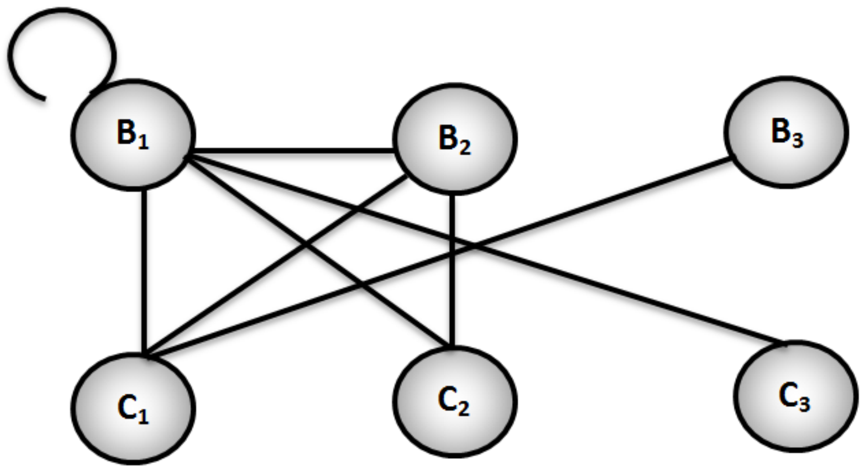


Figure 3.1 : The general form of the graph $\Gamma(Z_{p^3q})$

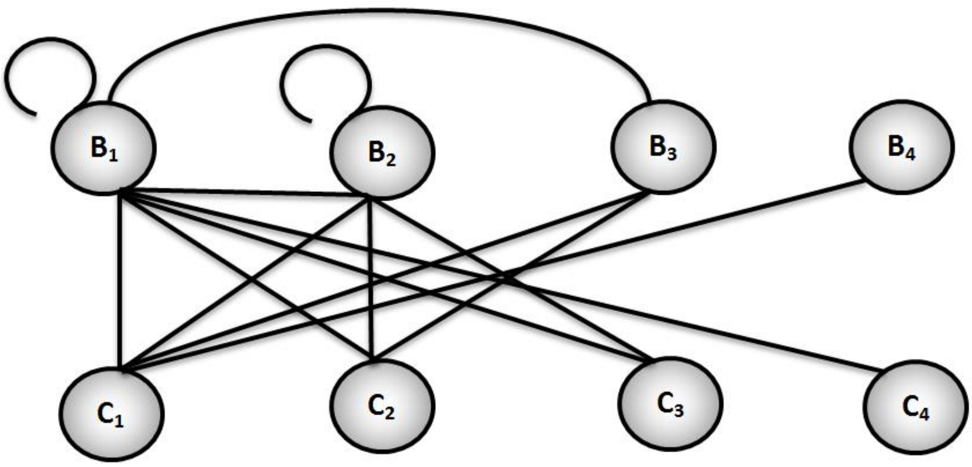


Figure 3.2 : The general form of the graph $\Gamma(Z_{p^4q})$

We can now give the general form of the graph $\Gamma(Z_{p^mq})$, as the following :

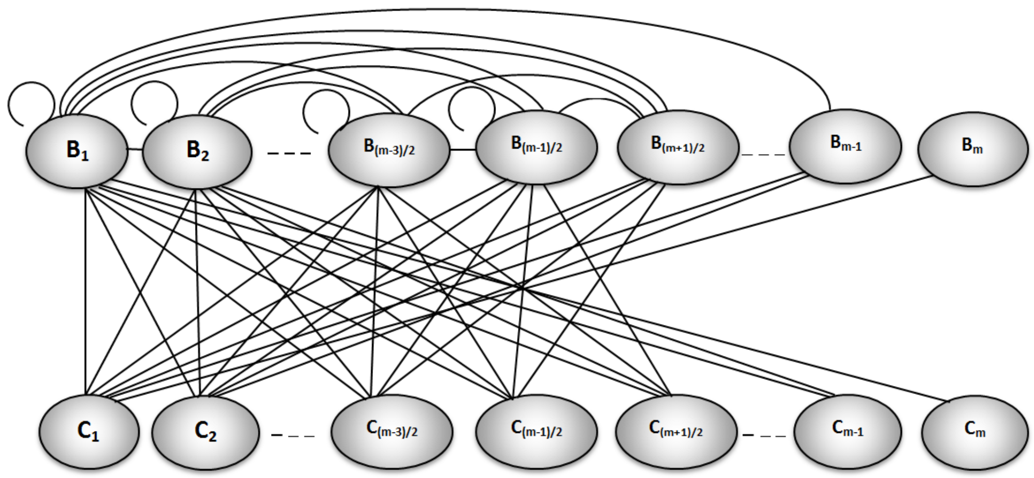


Figure 3.3 : The general form of the graph $\Gamma(Z_{p^m q})$, where m is an odd number with m≥5.

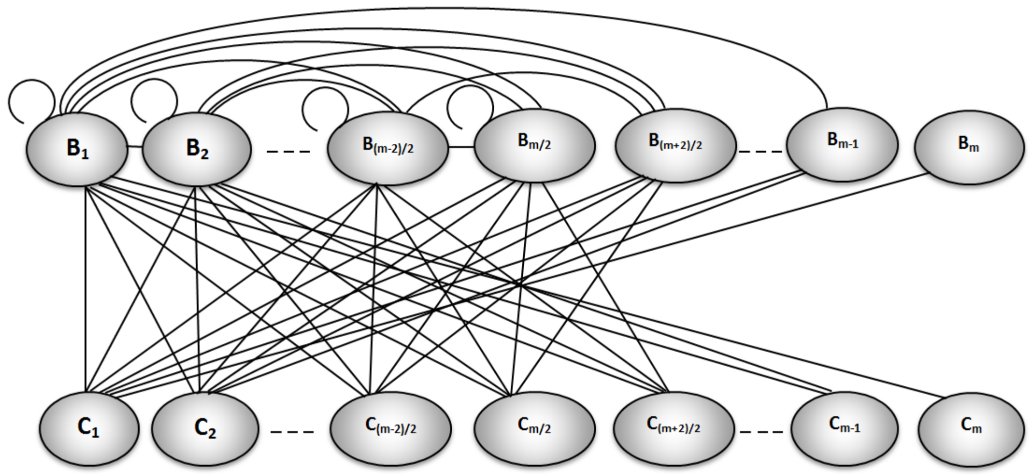


Figure 3.4 : The general form of the graph $\Gamma(Z_{p^m q})$, where m is an even number with m≥6.

<u>Lemma 3.5</u> [8, Proposition 3.2.] : Let $Z_{p^m q}$ be a ring of integers modulo $p^m q$. Then, $\mathrm{diam}(\Gamma(Z_{p^m q}))= 3$.

Now, we give the main result in this section.

<u>Theorem 3.6</u>: $H(\Gamma(Z_{p^m q});\ x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$, where

$a_0 = (p+q-1)\, p^{m-1} - 1$,

$a_1 = \frac{1}{2}\,[\, 2mq\,(p-1) - (m+1)\, p + m]\, p^{m-1} - \frac{1}{2}\, p^{\lfloor \frac{m}{2} \rfloor} + 1$,

$a_2 = \frac{1}{2}\,(p^2 + q^2 - 1)\, p^{2m-2} + \frac{1}{2}\,[(m-4)\, p - 2(m-1)\, pq + (2m-5)q - m + 5]\, p^{m-1} + \frac{1}{2}\, p^{\lfloor \frac{m}{2} \rfloor}$, and

$a_3 = (q-1)(p-1)\,(p^{2m-2} - p^{m-1})$.

Proof : By Lemma 3.5 we have $\mathrm{diam}(\Gamma(Z_{p^m q})) = 3$, then $H(\Gamma(Z_{p^m q});\ x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$, where $a_i = d(\Gamma(Z_{p^m q}), i)$, for i = 0,1,2,3 .

To find a_0, by Lemma 3.3 we have

$a_0 = d(\Gamma(Z_{p^m q}), 0) = |Z^*(Z_{p^m q})| = (p+q-1)\, p^{m-1} - 1$.

Now, to find a_1, let $x,y \in Z^*(Z_{p^m q})$ such that $d(x,y)=1$ (i.e. $xy=0$), hence by using Theorem 3.4 there are two cases :

Case 1 : $x\in B_i$ and $y\in B_j$ with $i+j \le m$, for some $1\le i,j\le m$, the same as the proof of Theorem 2.7, we get that there are m_1 edges in this case, where

$m_1=\frac{1}{2}[(m-1)\,p^m - m\,p^{m-1} - p^{\lfloor \frac{m}{2} \rfloor} + 2]$ . . . (*)

Case 2 : $x\in B_i$ and $y\in C_j$ with $i+j \le m+1$, for some $1\le i,j\le m$, this holds if and only if $1\le i\le m$ and $1\le j\le m-i+1$, because that $i+j \le m+1$ for any $1\le i\le m$ and $1\le j\le m-i+1$, so that $i+j>m+1$ in otherwise of this case , so that there are m_2 edges, where $m_2=\sum_{i=1}^{m}(|B_i|\sum_{j=1}^{m-i+1}|C_j|)$, and since $|B_i|=(p^i-p^{i-1})$ for $1\le i\le m$, then by Lemma 3.3, we get that

$m_2=\sum_{i=1}^{m}(p^i-p^{i-1})p^{m-i+1-1}(q-1)=\sum_{i=1}^{m}p^{i-1}(p-1)p^{m-i}(q-1)$

. $=\sum_{i=1}^{m}(p-1)(q-1)\,p^{m-1} = m\,(p-1)(q-1)\,p^{m-1}$ . . . (* *)

Now, from (*) and (**), we get that

$a_1=m_1+m_2=\frac{1}{2}(m-1)\,p^m-\frac{1}{2}m\,p^{m-1}-\frac{1}{2}p^{\lfloor \frac{m}{2} \rfloor}+1+m\,(p^m-p^{m-1})\,(q-1)$

$=\frac{1}{2}m\,p^m-\frac{1}{2}p^m-\frac{1}{2}m\,p^{m-1}-\frac{1}{2}p^{\lfloor \frac{m}{2} \rfloor}+1+m\,p^mq-m\,p^m-m\,p^{m-1}q+mp^{m-1}$

$=\frac{1}{2}[\,2mq\,(p-1)-(m+1)\,p+m]\,p^{m-1}-\frac{1}{2}p^{\lfloor \frac{m}{2} \rfloor}+1$

Now, to find a_i, for i=2,3 ,in the first, we shall find a_3 .

Let $x,y\in Z^*(Z_{p^mq})$ such that d(x,y)=3, then $x\in B_i$ and $y\in C_j$ for some $1\le i,j\le m$, in this case, we see that d(x,y)=3 if and only if i=m and $2\le j\le m$, because that $d(x,y)\le 2$ for any $1\le i\le m-1$ and $2\le j\le m$, also that d(x,y)=1 for $1\le i\le m$ and j=1, therefore the number of pairs of vertices that are distance three apart is $(|B_m|\sum_{j=2}^{m}|C_j|)$, i.e. $a_3=|B_m|\sum_{j=2}^{m}|C_j|$, since $|B_m|=(p^m-p^{m-1})$, then by Lemma 3.3, we get that :

$a_3=(p^m-p^{m-1})(q-1)\,(p^{m-1}-1)=(q-1)(p-1)\,(p^{2m-2}-p^{m-1})$.

Now, to find a_2 we shall use lemma 2.9 , that is :

$a_2=\frac{1}{2}a_0\,(a_0+1)-a_0-a_1-a_3=\frac{1}{2}a_0\,(a_0-1)-a_1-a_3$

$=\frac{1}{2}((p+q-1)\,p^{m-1}-1)\,((p+q-1)\,p^{m-1}-2)-[\frac{1}{2}(\,2mq\,(p-1)-(m+1)\,p+m)\,p^{m-1}$
$-\frac{1}{2}p^{\lfloor \frac{m}{2} \rfloor}+1]-(q-1)(p-1)\,(p^{2m-2}-p^{m-1})$

$=\frac{1}{2}(p^2+q^2-1)\,p^{2m-2}+\frac{1}{2}[(m-4)\,p-2(m-1)\,pq+(2m-5)\,q-m+5]\,p^{m-1}+\frac{1}{2}p^{\lfloor \frac{m}{2} \rfloor}$. ■

Corollary 3.7 : $W(\Gamma(Z_{p^mq}))=[p^2+q^2+3(\,pq-p-q\,)+2]\,p^{2m-2}+\frac{1}{2}[(m-3)p-2(m+1)\,pq+2(m-2)q\,]\,p^{m-1}+\frac{1}{2}p^{\lfloor \frac{m}{2} \rfloor}+1$.

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