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On a family of holomorphic functions involving linear operator

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Abstract

The purpose of the present article is to study and introduce the family of holomorphic univalent functions defined on the open unit disk incluing the linear operators reduced by using the convolution concept between two familiar operators introduced by many authors and also studied in various families generalized by starlike and convex functions. Also, many mathematicians considered these families, our study deals with Bord distribution series, and we obtained many interesting geometric properties like the coefficient bounds of functions belonging to our family by proving the main result in characterization theorem, we obtained also the bounds of the derivative of the operators $\mathcal{T}(\lambda, \mathrm{w})$ by proving the distortion theorem ,the extreme points also take into consideration and found them, the radii convexity and starlikeness are pointed out of functions belong to this family .

Keywords:
Linear operator, Holomorphic functions, Univalent functions, Starlike functions, Radius problems.

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1. INTRODUCTION

Let M be a class of functions with the following form

$$f(w) = w + \sum_{\mu=2}^{\infty} a_\mu w^\mu , \qquad (1.1)$$

which are holomorphic and univalent in open unit disk; $\Omega = \{w \in C: |w|<1\}$.

A function $f \in M$ is called starlike of complex order

$$R\left\{1 + \frac{1}{t}\left(\frac{wf'(w)}{f(w)} - 1\right)\right\} > \propto , \quad (0 \leq \propto < 1 \text{ and } t \in \mathbb{C}\backslash\{0\}) \qquad (1.2)$$

A discrete random variable y which has a Borel-distribution if it takes the standards 1,2,3,… with the probabilities

$$\frac{e^{-1}}{1!}, \frac{2\lambda e^{-2\lambda}}{2!}, \frac{9\lambda^2 e^{-3\lambda}}{3!}, \ldots$$

Where y is the parameter, hence Prob $(y = b) = \frac{(\lambda b)^{b-1} e^{-\lambda b}}{b!}$. $(b \epsilon N)$

The following power series using probabilities from the Borel

$$\mathcal{T}(\lambda, w) = w + \sum_{\mu=2}^{\infty} L_{\lambda,\mu} w^\mu, \quad (w \in \Omega; 0 < \lambda \leq 1) \qquad (1.3)$$

Where

$$L_{\lambda,\mu} = \frac{(\lambda(\mu-1))^{\mu-2} e^{-\lambda(\mu-1)}}{(\mu-1)!}, \quad (2 \leq \mu < \infty).$$

Now, we introduce a linear operator $\mathcal{T}(\lambda, w): M \to M$ defined as

$$\mathcal{T}(\lambda, w) * f(w) = w + \sum_{\mu=2}^{\infty} L_{\lambda,\mu} a_\mu w^\mu , \quad (w \in \Omega; 0 < \lambda \leq 1). \qquad (1.4)$$

Motivated by many authors studied the several classes associated with many distribution series like logarthemtic distribution, Binomial distribution and zeta distribution (see for example. [1],[2],[3],[4],[6],[8])

Here, we have obtained some results and give conditions for the functions belonging in our classes.

Definition 1. $f(w) \in M$ supposedly in $\mathcal{P}(t, \propto)$ if the following condition holds:

$$R\left\{1+\frac{1}{t}\left(\frac{wf'(w)}{f(w)}-1\right)\right\} > \propto, \quad (0 \leq \propto < 1 \text{ and } t \in \mathbb{C}\backslash\{0\}). \qquad (1.5)$$

Moved by the work of Srivastava and Gaboury [7],Juma and Darus[5],and we investigate some geometric properties of the current function class such as coefficients-estimate, distortion bounds, Radius and the extreme points.

2. MAIN RESULT

Theorem 2.1. For $0 \leq \propto < 1, t \in \mathbb{C}\backslash\{0\}$ and if $f(w) \in M$ complies with the following conditions

$\sum_{\mu=2}^{\infty}(\mu - \propto |t|)L_{\lambda,\mu}\left|a_\mu\right| \leq 1 - \propto |t|$, then $f \in \mathcal{P}(t, \propto)$ (2.1)

Proof. Imagine that (2.1) satisfy, we have

$$C(w)=\frac{\frac{w\mathcal{T}'(\lambda,w)f(w)}{t\mathcal{T}(\lambda,w)f(w)}-\left(\frac{1}{t}+\propto-1\right)-1}{\frac{w\mathcal{T}'(\lambda,w)f(w)}{t\mathcal{T}(\lambda,w)f(w)}-\left(\frac{1}{t}+\propto-1\right)+1}$$

As a result, it is enough to show that $|C(w)| < 1$ that is

$$\boldsymbol{C(w)}=\left|\frac{\frac{w\mathcal{T}'(\lambda,w)f(w)}{t\mathcal{T}(\lambda,w)f(w)}-\left(\frac{1}{t}+\propto-1\right)-1}{\frac{w\mathcal{T}'(\lambda,w)f(w)}{t\mathcal{T}(\lambda,w)f(w)}-\left(\frac{1}{t}+\propto-1\right)+1}\right|$$

$$=\left|\frac{\propto tw+\sum_{\mu=2}^{\infty}L_{\lambda,\mu}(\mu-1-\propto t)a_\mu w^\mu}{(2-\propto)tw+\sum_{\mu=2}^{\infty}L_{\lambda,\mu}(\mu+1-\propto t)a_\mu w^\mu}\right|,$$

thus by using (2.1) we have

$$\boldsymbol{C(w)} \leq$$

$$\leq \frac{\propto|t||w|+\sum_{\mu=2}^{\infty}L_{\lambda,\mu}(\mu-1-\propto|t|)|a_\mu||w^\mu|}{(2-\propto)|t||w|-\sum_{\mu=2}^{\infty}L_{\lambda,\mu}(\mu+1-\propto|t|)|a_\mu||w^\mu|}$$

$$< \frac{\propto|t|+\sum_{\mu=2}^{\infty}L_{\lambda,\mu}(\mu-1-\propto|t|)a_\mu}{(2-\propto)|t|-\sum_{\mu=2}^{\infty}L_{\lambda,\mu}(\mu+1-\propto|t|)a_\mu} \leq \mathbf{1},$$

and the proof is complete.

Definition 2. Let T represents the subclass of M of the kind

$$f(w)=w-\sum_{\mu=2}^{\infty}a_\mu w^\mu, \quad a_\mu \geq 0 \qquad (2.2)$$

if $f \in T$ is given by (2.2) then we have

$$\mathcal{T}(\lambda,w)*f(w)=w-\sum_{\mu=2}^{\infty}L_{\lambda,\mu}a_\mu w^\mu.$$

Therefore, we take $T\mathcal{P}(t,\propto)=\mathcal{P}(t,\propto)\cap T$.

Theorem 2.2. A function f of the form (2.2) which are from the class $T\mathcal{P}(t,\propto) \leftrightarrow$

$$\sum_{\mu=2}^{\infty}(\mu-\propto|t|)L_{\lambda,\mu} \leq 1-\propto|t|$$

Proof. Since $T\mathcal{P}(t,\propto) \subset \mathcal{P}(t,\propto)$. Then $f \in T\mathcal{P}(t,\propto)$ by Theorem (2.1).
Conversely, since $f \in T\mathcal{P}(t,\propto)$,then

$$R\left\{1+\frac{1}{t}\left(\frac{w\mathcal{T}'(\lambda,w)f(w)}{\mathcal{T}(\lambda,w)f(w)}-1\right)\right\}$$

$$=R\left\{\frac{t-\sum_{\mu=2}^{\infty}(t+\mu-1)L_{\lambda,\mu}a_\mu}{t(1-\sum_{\mu=2}^{\infty}L_{\lambda,\mu}a_\mu)}\right\} > \propto.$$

We select w's value. on the real axis. Let $w \to 1^{-1}$ though real values, we discover inequality (2.1).

Corollary 2.1.let the function $f(w)$ described by (2.2) be in the class $T\mathcal{P}(t,\propto)$ then

$$\left|a_\mu\right| \leq \frac{1-\propto|t|}{(\mu-\propto|t|)L_{\lambda,\mu}}, \qquad \mu \geq 2$$

3. DISTORTION BOUNDS

Theorem 3. *For* $0 \leq \propto < 1, t \in \mathbb{C}\backslash\{0\}$, and suppose $f(w)$ belongs to in the class $T\mathcal{P}(t,\propto)$ therefore,

$$s-\frac{(1-\propto)|t|^2}{Re(t)}s^2\sum_{\mu=2}^{\infty}\frac{1}{L_{\lambda,\mu}\left(\mu-1+\frac{(1-\propto)|t^2|}{Re(t)}\right)} \leq$$

$$|f(w)| \leq s+\frac{(1-\propto)|t|^2}{Re(t)}s^2\sum_{\mu=2}^{\infty}\frac{1}{L_{\lambda,\mu}\left(\mu-1+\frac{(1-\propto)|t^2|}{Re(t)}\right)},$$

$(|w|=s<1) \qquad (3.1)$

$$1-\frac{(1-\propto)|t|^2}{Re(t)}s\sum_{\mu=2}^{\infty}\frac{1}{L_{\lambda,\mu}\left(\mu-1+\frac{(1-\propto)|t^2|}{Re(t)}\right)} \leq$$

$$|f'(w)| \leq 1+\frac{(1-\propto)|t|^2}{Re(t)}s\sum_{\mu=2}^{\infty}\frac{1}{L_{\lambda,\mu}\left(\mu-1+\frac{(1-\propto)|t^2|}{Re(t)}\right)},$$

$(|w|=s<1). \qquad (3.2)$

Proof. Belong to M. In virtue of Theorem 2.1, we get

$$|f(w)| \geq |w| - \sum_{\mu=2}^{\infty}|a_\mu||w^\mu| \geq$$

$$\geq s - \frac{(1-\propto)|t|^2}{Re(t)} s^2 \sum_{\mu=2}^{\infty} \frac{1}{L_{\lambda,\mu}\left(\mu-1+\frac{(1-\propto)|t^2|}{Re(t)}\right)}$$

and

$$|f(w)| \leq |w| + \sum_{\mu=2}^{\infty}|a_\mu||w^\mu| \leq$$

$$\leq s + \frac{(1-\propto)|t|^2}{Re(t)} s^2 \sum_{\mu=2}^{\infty} \frac{1}{L_{\lambda,\mu}\left(\mu-1+\frac{(1-\propto)|t^2|}{Re(t)}\right)},$$

$(|w| = s < 1)$.

Also, from (1.1), there are

$$|f'(w)| \geq 1 - \sum_{\mu=2}^{\infty} \mu|a_\mu||w^{\mu-1}| \geq$$

$$\geq 1 - \frac{(1-\propto)|t|^2}{Re(t)} s \sum_{\mu=2}^{\infty} \frac{1}{L_{\lambda,\mu}\left(\mu-1+\frac{(1-\propto)|t^2|}{Re(t)}\right)}$$

and

$$|f'(w)| \leq 1 + \sum_{\mu=2}^{\infty} \mu|a_\mu||w^{\mu-1}| \leq$$

$$\leq 1 + \frac{(1-\propto)|t|^2}{Re(t)} s \sum_{\mu=2}^{\infty} \frac{1}{L_{\lambda,\mu}\left(\mu-1+\frac{(1-\propto)|t^2|}{Re(t)}\right)},$$

$(|w| = s < 1)$.

Thus, we obtained the required results.

4. EXTREME POINTS

We consider the subclass $\hat{\mathcal{P}}(t,\propto)$ of the class $\mathcal{P}(t,\propto)$ in this section, which consists of all the functions $f(w) \in M$ of the form (1.1), and we satisfy (2.1). The extreme points of the subclass $\hat{\mathcal{P}}(t,\propto)$ are defined by the following theorem.

Theorem 4. Let $\quad f_1(w) = w \qquad\qquad (4.1)$

and

$$f_\mu(w) = w + \frac{1}{(\mu-\propto|t|)|L_{\lambda,\mu}|} w^\mu, \ (\mu \neq 1).$$

Then $f \in \hat{\mathcal{P}}(t,\propto)$ if, and only if

$f(w) = \sum_{\mu=1}^{\infty} \eta_\mu f_\mu(w) \quad (\eta_\mu > 0; \sum_{\mu=1}^{\infty} \eta_\mu = 1)$
(4.3)

Proof. Let $f \in \hat{\mathcal{P}}(t,\propto)$. Then in virtue of (2.1). we can set

$$\eta_\mu = (\mu-\propto |t|)\frac{L_{\lambda,\mu}}{(1-\propto|t|)}, \ (\mu \neq 1). \qquad (4.4)$$

Which results in our (4.3).
Conversely, let

$$f(w) = \sum_{\mu=1}^{\infty} \eta_\mu f_\mu(w)$$

$$= w + \sum_{\mu=1}^{\infty} \eta_\mu \frac{1}{(\mu-\propto|t|)|L_{\lambda,\mu}|} w^\mu.$$

Then

$$\sum_{\mu=1}^{\infty}(\mu-\propto |t|)L_{\lambda,\mu}.$$

$$\eta_\mu \frac{1}{(\mu-\propto|t|)|L_{\lambda,\mu}|} =$$

$$1-\propto |t| \sum_{\mu=2}^{\infty} \eta_\mu = (1-\propto |t|)(1-\eta_1).$$

Therefore, we have $f \in \hat{\mathcal{P}}(t,\propto)$.

5. RADII OF STARLIKENESS AND CONVEXITY

Theorem 5.1. If $f \in T\mathcal{P}(t,\propto)$ of the form (2.2), then f is holomorphically starlike from order $\propto$, $(0 \leq\propto< 1)$ in the disk $|w| < r, (\propto, t)$ where

$$r_2(\propto, t) = \inf_{\mu\geq 2}\left[\frac{(\mu-\propto|t|)L_{\lambda,\mu}(1-\propto)}{(1-\propto|t|)(\mu-\propto)}\right]^{\frac{1}{\mu-1}} \qquad (5.1)$$

Proof. By a computation, we have

$$\left|\frac{wf'(w)}{f(w)} - 1\right| = \left|\frac{-\sum_{\mu=2}^{\infty}(\mu-1)a_\mu w^{\mu-1}}{1-\sum_{\mu=2}^{\infty} a_\mu w^{\mu-1}}\right| \leq$$

$$\frac{\sum_{\mu=2}^{\infty}(\mu-1)a_\mu|w|^{\mu-1}}{1-\sum_{m=2}^{\infty} a_\mu|w|^{\mu-1}}.$$

Thus f is starlike of order $\propto$ if

$$\sum_{\mu=2}^{\infty} \frac{(\mu-\propto)}{1-\propto} a_\mu|w|^{\mu-1} \leq 1. \qquad (5.2)$$

Since $f \in T\mathcal{P}(t,\propto)$,we have

$$\sum_{\mu=2}^{\infty} \frac{(\mu-\propto|t|)L_{\lambda,\mu}}{1-\propto|t|} \leq 1. \qquad (5.3)$$

Now, (5.3) holds if

$$\frac{(\mu-\propto)}{1-\propto}|w|^{\mu-1} \leq \frac{(\mu-\propto|t|)L_{\lambda,\mu}}{1-\propto|t|},$$

That is, if

$$|w| \leq \left[\frac{(\mu - \propto |t|)L_{\lambda,\mu}(1-\propto)}{(1-\propto|t|)(\mu-\propto)}\right]^{\frac{1}{\mu-1}} \qquad (5.4)$$

Theorem 5.2. If $f \in \mathcal{TP}(t,\propto)$ of the form (2.2),then f is holomorphically convex from order $\propto$, $(0 \leq \propto < 1)$ in the disk $|w| < r,(\propto, t)$ where

$$r_2(\propto, t) = \inf_{\mu\geq 2}\left[\frac{(\mu-\propto|t|)L_{\lambda,\mu}(1-\propto)}{(1-\propto|t|)(\mu(\mu-\propto))}\right]^{\frac{1}{\mu-1}} \qquad (5.5)$$

Proof. By a computation, we have

$$\left|\frac{wf''(w)}{f'(w)}\right| = \left|\frac{-\sum_{\mu=2}^{\infty}\mu(\mu-1)a_\mu w^{\mu-1}}{1-\sum_{\mu=2}^{\infty}\mu a_\mu w^{\mu-1}}\right| \leq$$

$$\frac{\sum_{\mu=2}^{\infty}\mu(\mu-1)a_\mu|w|^{\mu-1}}{1-\sum_{\mu=2}^{\infty}\mu a_\mu|w|^{\mu-1}}$$

Thus f is convex of order $\propto$ if

$$\sum_{\mu=2}^{\infty}\frac{\mu(\mu-\propto)}{1-\propto}a_\mu|w|^{\mu-1} \leq 1.$$

(5.6)

Since $f \in \mathcal{TP}(t,\propto)$,we have

$$\sum_{\mu=2}^{\infty}\frac{(\mu-\propto|t|)L_{\lambda,\mu}}{1-\propto|t|} \leq 1.$$

(5.7)

Now, (5.7) holds if

$$|w| \leq \inf_{\mu\geq 2}\left[\frac{(\mu-\propto|t|)L_{\lambda,\mu}(1-\propto)}{(1-\propto|t|)(\mu(\mu-\propto))}\right]^{\frac{1}{\mu-1}}$$

(5.8)

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الملخص

الغرض من هذا البحث هو دراسة و تقديم كاملة من الدوال التحليلية احادية التكافؤ المعرفة على قرص الموحدة و باستخدام مؤثر خطي تم استنتاجه من خلال مفهوم الالتفافات بين مؤثرين خطين المقدمة من قبل عدد من الباحثين في هذا المجال بالإضافة الى دراسة لتعميم مفهوم الدوال النجمية و المحدبة دراستنا ترتبط مع سلسلة التوزيع و الحصول على عدد من الخصائص المهمة الهندسية مثل قيود المعاملات الدوال التي تنمي الى هذا العائلة و كذلك نظريات التشويه و ايجاد النقاط المتطرفة بالإضافة الى انصاف الاقطار النجمية و المحدبة للدوال التي تنتمي الى هذه العائلة.

الكلمات المفتاحية : مؤثر خطي, دوال تحليلية, دوال احاديه التكافؤ, الدوال النجمية مسائل انصاف الاقطار.