On N–Flat Rings

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ABSTRACT

Let I be a right ideal of R, then R / I is a right N–flat if and only if for each a $\in$ I, there exists b $\in$ I and a positive integer n such that $a^n \neq 0$ and $a^n = ba^n$. In this paper, we first give and develop various properties of right N-flat rings, by which, many of the known results are extended. Also, we study the relations between such rings and regular, π-biregular ring.

Key word: N-flat rings, weakly continuous rings, biregular rings.

حول الحلقات المسطحة من النمط – N

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الملخص

ليكن I مثالي أيمن في R , فان R/I حلقة مسطحة يمنى من النمط –N إذا وفقط إذا لكل a∈I يوجد b∈I وعدد صحيح موجب n بحيث ان an≠0 و an=ban في هذا البحث أعطينا أولا خواصا متنوعة للحلقات المسطحة من النمط – N , كما قمنا بتطوير عدد من النتائج المعروفة. كذلك درسنا العلاقة بين تلك الحلقات والحلقات المنتظمة والحلقات المنتظمة ثنائيا من النمط – π .

الكلمات المفتاحية: الحلقات المسطحة من النمط – N , الحلقات المستمرة بضعف, الحلقات المنتظمة الثنائية.

1. Introduction:

Throughout this paper R is associative ring with identity, and R-module is unital. For a $\in$ R, r(a) and l(a) denote the right annihilator and the left annihilator of a, respectively. We write J(R), P(R), Y(R) (Z(R)) and N(R) for the Jacobson radical, the prime radical, the right (left) singular ideal and the set of nilpotent elements of R, respectively.

(1) A ring R is called a right **SF-ring** [8] if every simple right R-module is flat. (2) A ring R is said to be right (left) **quasi-duo** [11] if every maximal right (left) ideal is a two-sided ideal of R. (3) A ring R is said to be **reversible** [3] if ab = 0 implies ba = 0, a,b $\in$ R. (4) A ring R is called **reduced** if contains no non-zero nilpotent elements. (5) A ring R is called **Von Neumann (strongly** resp.) regular provided that for every a $\in$ R there exists b$\in$R such that a = aba (a = a^2b, resp.). (6) A ring R is called **biregular** [7] if for any a R, RaR is generated by a central idempotent. (7) A ring R is said to be **π-biregular** [7] if for any a R, Ra^nR is generated by a central idempotent for some positive integer n. (8) A ring R is called right (left) **Kasch ring** [4] if every maximal right (left) ideal of R is a right (left) annihilator. (9) A ring R is called **2-primal** if the set of nilpotent elements of the ring coincides with the prime radical.

2. Simple N–flat:

We introduce the notion of a right N–flat with some of their basic properties. We also give some relation between right N–flat rings and other rings.

Definition 2.1: Let I be a right (left) ideal of R. Then R / I is a right (left) N–flat if and only if for each $a \in I$, there exists $b \in I$ and a positive integer n such that $a^n \neq 0$ and $a^n = ba^n$ ($a^n = a^n b$). ■

The following example illustrates the above definition.

Examples:

(1) Let Z_{10} be the ring of integers modulo 10 and I = {0, 2, 4, 6, 8}, J = {0, 5}. Then Z_{10}/I and Z_{10} / J are N – flat.
(2) Let Z_9 be the ring of integers modulo 9 and K = {0, 3, 6}. Then Z_9/K in not N – flat.

Remark (1): Every SF – ring is simple N – flat. ■

Proposition 2.2: Let R be a ring whose every simple right R – module is right N – flat. Then,
(1) Every left non – zero divisor element is a right invertible.
(2) Z (R) $\subseteq$ J(R).

Proof: (1) Let $a \neq 0$ be a left non – zero divisor, if $aR \neq R$, then there exists a maximal right ideal M of R containing aR. Since $a \in aR$ □□M, and R / M is right N – flat, then there exists a positive integer n and $b \in M$ such that $a^n \neq 0$ and $a^n = ba^n$ which implies $(1–b)\, a^n = 0$. Since a is left non – zero divisor, then (1–b) = 0, and we get $b = 1 \in M$ which is a contradiction. Thus aR = R, and hence a is right invertible.
(2) Let $z \in Z(R)$, then for any $r \in R$, we have $l(1 – r z) = 0$, which implies that (1–rz) is right invertible, so that $z \in J(R)$. Therefore $Z(R) \subseteq J(R)$. ■

Proposition 2.3 : If R is a ring whose every simple right R – module is right N – flat and R has a finite number of maximal right ideals whose product is contained in J(R), then Z (R) = J(R) = 0 .

Proof: Let $M_1, M_2, \ldots, M_m$ be maximal right ideals of R such that $M_1M_2\ldots M_m \subseteq J(R)$.
First, suppose that J(R) is non – zero reduced. If $x \in J(R)$, and since $x \in M_m$ and R/M_m is N–flat, then there exist a positive integer n_m and $y_m \in M_m$ such that $x^{n_m} = y_m x^{n_m}$, which implies that $1 - y_m \in r(x^{n_m})$. Since J(R) is reduced and $x \in J(R)$, then $r(x) = r(x^{n_m})$, thus $1 - y_m \in r(x^{n_m}) = r(x)$. Therefore $x = y_m x$. Since $y_m x \in J(R) \subseteq M_{m-1}$ and R/M_{m-1} is N – flat, there exist a positive integer n_{m-1} and $y_{m-1} \in M_{m-1}$ such that $x^{n_{m-1}} = y_{m-1} x^{n_{m-1}}$ and we get $x = y_{m-1} x$, and so on.
Finally, we have $y_i \in M_i$, $1 \leq i \leq m$, such that
$y_1\, y_2 \ldots y_{m-1}\, y_m \in M_1\, M_2 \ldots M_m \subseteq J(R)$ and $x = y_1\, y_2 \ldots y_{m-1}\, y_m x$.
Now $z(1 - y_1\, y_2 \ldots y_m) = 1$ for some $z \in R$ which yields $x = 1x = z\,(1 - y_1\, y_2 \ldots y_m)\, x = 0$, which is a contradiction.
Now suppose that J(R) is not reduced. Then there exists $0 \neq a \in J(R)$ such that $a^2 = 0$. Since $a \in J(R) \subseteq M_m$ and R / M_m is N – flat, then $a = b_m a$ for some $b_m \in M_m$. Since $b_m a \in J(R) \subseteq M_{m-1}$ and R / M_{m-1} is N–flat, then $a = b_m a = b_{m-1}\, b_m a$ for some $b_{m-1} \in M_{m-1}$ and so on.
Finally we have $b_i \in M_i$, $1 \leq i \leq m$, such that
$b_1\, b_2 \ldots b_{m-1}\, b_m \in M_1\, M_2 \ldots M_{m-1}\, M_m \subseteq J(R)$ and $a = b_1\, b_2 \ldots b_{m-1}\, b_m a$.
Now $u(1-b_1\, b_2 \ldots b_m) = 1$ for some $u \in R$ which yields
$a = 1a = u(1-b_1 \ldots b_m)a = 0$. Thus J(R)=0 and by Proposition 2.2 $Z(R) \subseteq J(R)$, thus Z(R)= 0. ■

Recall that a ring R is right (left) **weakly continuous** if J(R)=Y(R)(J(R)=Z(R)), R / J(R) is regular and idempotent can be left module J(R). Clearly every regular ring is right (left) weakly continuous .

Corollary 2.4 : Let R be a left weakly continuous, whose every simple right R – module is N – flat and R has a finite number of maximal right ideals whose product is contained in J(R). Then R is regular. ■

Lemma 2.5 : [2]**:** A ring R has zero prime radical if and only if it contains no – nonzero nilpotent ideal. ■

Theorem 2.6: Let R be a semi–prime 2–primal ring whose every simple right R–module is N–flat. Then R is biregular.

Proof : Let $0 \neq a \in R$ such that $a^2 = 0$. Thus, $a \in P(R)$. Now, since R is semi–prime ring then R has no non – zero nilpotent ideal, and by Lemma2.5, P(R) = 0, so a = 0 and hence R is reduced.
Now, for any $0 \neq a \in R$, r(RaR) = l(RaR) = l(aR) = r(aR)=r(a). If E = RaR + r(a), then E = RaR $\oplus$ r(RaR) [since RaR $\cap$ r(RaR) = 0].
Suppose that $E \neq R$. Let M be a maximal right ideal of R. Since R/M is N – flat and $a \in M$, there exists $b \in M$ and a positive integer n , such that $a^n \neq 0$ and $a^n = ba^n$.
Now, $1–b \in l\,(a^n) = r\,(a^n) = r(a) \subseteq M$ which implies that $l \in M$ a contradiction. We have proved that R= E = RaR $\oplus$ r (RaR). Since every idempotent in reduced ring is central, then RaR is generated by a central idempotent. ■

Lemma 2.7. [11]: If R is a right quasi-duo with J(R) = 0 , then R is reduced. ■
We now consider other condition for right simple N-flat to be biregular.

Theorem 2.8: If R is right quasi duo ring whose every simple right R – module is N – flat and R has a finite number of maximal right ideals whose product is contained in J(R), then R is biregular.

Proof : By Proposition 2.3, J(R) = 0. Since R is right quasi-duo, then R is reduced by Lemma 2.7. The proof of R being biregular is similar to that of Theorem 2.6. ■

Remark (2) [5]: If M is an essential right ideal, then R_R / M can not be projective.■

We consider the condition (*) : R satisfies $l\,(a) \subseteq r\,(a)$ for any $a \in R$.

We begin with a property of rings whose simple right R–module are either N–flat or projective.

Theorem 2.9: Let R b a ring satisfy condition (*). If every simple right R – module is either N – flat or projective, then Z(R) $\cap$ Y(R) = 0 .

Proof: Let us first suppose that Z(R) $\cap$ Y(R) is non–zero reduced ideal of R. If $0 \neq x \in Z(R) \cap Y(R)$, r(x) is essential right ideal of R and $xR \cap r(x) \neq 0$. Let $a \in R$ such that $0 \neq xa \in r(x)$. Since Z(R) $\cap$ Y(R) is reduced and $xax \in Z(R) \cap Y(R)$, then $(xax)^2=0$ which implies xax=0. Therefore $(xa)^2 = 0$, which yields xa = 0, a contradiction.
Now, suppose that $Z(R) \cap Y(R) \neq 0$, then there exists $0 \neq y \in Z(R) \cap Y(R)$ such that $y^2= 0$.
We will prove that RyR + r(y) = R.
If not, let M be a maximal right ideal containing RyR + r(y). Since r(y) is essential right ideal then R / M can not be projective by Remark (2), whence it is N – flat. Since R / M is N – flat, then there exist $d \in M$ and a positive integer n such that $y^n \neq 0$ and $y^n = dy^n$

, Since $y^2 = 0$, then n = 1, so that y = dy and we get $1–d \in l(y) \subseteq r(y) \subseteq M$ and $l \in M$. Whence M = R, contradicting the maximally of M. Therefore R=RyR + r(y) .
Now, 1 = u + z , $u \in RyR$, $z \in r(y)$ which implies that y = yu . Since $u \in Z(R)$ and $Ry \cap l(u) = 0$ then y = 0 a contradiction. We have proved that $Z(R) \cap Y(R) = 0$. ■

Corollary 2.10: Let R be a right weakly continuous satisfy condition (*). If every simple right R–module is N–flat or projective, then $Z(R) \cap J(R) = 0$. ■

Corollary 2.11: Let R be weakly continuous ring satisfying condition (*). If every simple right R–module is N–flat or projective, then R is regular. ■

Proposition 2.12: Let R be a semi–prime ring satisfying condition (*), whose every simple right R–modules is either N–flat or projective. Then R is left non–singular.

Proof : Suppose that $Z(R) \neq 0$. Then there exists, $0 \neq z \in Z(R)$ such that $z^2 = 0$. Set L = RzR + r(z) . Let K be a complement right ideal of R, then $E = L \oplus K$ is an essential right ideal of R .
Then $KRzR \subseteq K \cap RzR \subseteq K \cap L = 0$ implies that $(RzRK)^2 = 0$. Since R is semi–prime then RzRK = 0, which yields $K \subseteq r(z) \subseteq L$. Whence $K = K \cap L = 0$. This shows that E = L is an essential right ideal of R .
Now suppose that $L \neq R$. Let M be a maximal right ideal of R containing L. Then R / M is N–flat, and there exists $u \in M$ and a positive integer n such that $z^n \neq 0$ and $z^n = uz^n$ which yields n=1 and $1–u \in l(z) \subseteq r(z) \subseteq M$. Thus $1 \in M$, contradicting $M \neq R$.
Therefore L = R and 1 = s + t where $s \in RzR$, $t \in r(z)$ and we have z = zs + zt = zs . Now $Rz \cap l(s) = 0$ implies that z = 0. This is a contradiction, thus R is left non – singular. ■

Applying Proposition 2.12 we get the next result.

Corollary 2.13: If R is a semi–prime left weakly continuous ring satisfying condition (*) such that every simply right R–module is either N–flat or projective, then R is regular. ■

Recall that a ring R is called a FGP-injective ring [1] if, for any $0 \neq a \in R$, there exists $0 \neq c \in R$ such that $0 \neq ac = ca$ and any right R− homomorphism from acR to R extends to an endomorphism of R.

Lemma 2.14 [9]: If Y(R) = 0 and satisfy condition (*), then R is reduced. ■

The following result is given in [1]

Lemma 2 .15: If R is a right Kasch FGP-injective ring, then J(R)=Y(R)=Z(R). ■

Comparing Theorem 2.9 with Lemma 2.15 , we ask the following question:

Question: Is a ring satisfying condition (*) whose every simple right R–module is either N–flat or projective strongly regular ring ?

Theorem 2.16: Let R be a right Kasch and right FGP-injective ring satisfying condition (*) and whose every simple right R–module is N–flat or projective. Then R is strongly regular .

Proof: Since R is right Kasch, right FGP-injective ring, then Z(R)=J(R)=Y(R) by Lemma 2.15 and $Z(R) \cap Y(R)=0$ by Theorem 2.9, which implies Z(R)=Y(R)=0. Therefore R is reduced by Lemma 2.14. Let $0 \neq a \in R$, we shall prove that aR+r(a)=R. If not, then there exists a maximal right ideal M containing aR + r(a). Since R is a right

Kasch ring, then there exists $b \in R$ such that $M = r(b)$. Let $x = ab + y$, where $b \in R$, $y \in r(a)$. So $x \in aR + r(a) \subseteq r(b)$ and $bx = b(ab+y) = 0$, since $by = 0$, then $bab=0$. But R is reduced so we have $ab = ba = 0$, which implies $b \in r(a) \subseteq r(b)$, therefore $b^2=0$, since R is reduced then $b= 0$, which is contradiction. So that $aR+r(a)=R$ and therefore, R is strongly regular ring. ■

3. Rings Whose Simple Singular R – Modules are N–Flat

In this section, we give further properties of rings for which every simple singular R–modules are N–flat.

Theorem 3.1: If R is a ring whose every simple singular right R–module is N–flat and satisfying condition (*), then $J(R) \cap Y(R) = 0$.

Proof : If $J(R) \cap Y(R) \neq 0$,there exists an element $0 \neq a \in J(R) \cap Y(R)$ such that $a^2 = 0$. If $r(a) + RaR \neq R$, there exists a maximal right ideal M of R containing $r(a)+RaR$. Since $a \in Y(R)$, then $r(a)$ is an essential and so M must be essential. By assumption, the simple singular right R–module R / M is N–flat. Thus there exists a positive integer n and $b \in M$ such that $a^n \neq 0$ and $a^n = ba^n$. Since $a^2 = 0$, then $n = 1$, and therefore $a = ba$ which implies that $1-b \in l(a) \subseteq r(a) \subseteq M$. Thus $1 \in M$, which is a contradiction. This proves that $r(a) + RaR = R$, and hence $a = ad$ for some $d \in RaR \subseteq J(R)$.

Thus $(1-d)$ is invertible and we get $a = 0$, which is the required contradiction. Therefore $J(R) \cap Y(R) = 0$. ■

Theorem 3.2: If R is a ring satisfying condition (*) and whose every simple singular right R–module is N – flat, then $J(R) = 0$ if and only if J(R) is a reduced ideal of R.

Proof : Suppose that J(R) is reduced. If for any $a \in J(R)$, then set $L=aR+r(a)$. If $L=R$, then $1 = ab + c$, for some $b \in R$ and $c \in r(a)$, which implies that $a = a^2b$. Since $a \in J(R)$, then $a - aba \in J(R)$ and $(a - aba)^2=0$ which yields $a = aba$.
Therefore $a = ae$, where $e = ba$ is idempotent. Since J(R) can not contain a non–zero idempotent, then $a = 0$.
If $L \neq R$, then there exists a right ideal M of R such that $L \oplus M$ is an essential right ideal of R.
We claim that $L \oplus M = R$. If not, there is a maximal essential right ideal K of R containing $L \oplus M$. By assumption, the simple singular right R–module R / K is N–flat. Since J(R) contains no non–zero nilpotent elements and $a \in J(R)$, then $a \in K$ and $a^n = da^n$ for some $d \in K$, $a^n \neq 0$ and a positive integer n.
Now $(1-d) \in l(a^n) = r(a^n) = r(a) \subseteq K$. Which implies that $1 \in K$, contradicting that K is maximal. This shows that $L \oplus M = R$.
Then $aR+r(a) = eR$ with $e^2 = e \in R$. So $a^2 = a^2e = aea = abaa = ba^2$, for some $b \in R$. But $a \in J(R)$, thus $a = 0$ by the proceeding proof. This proves that if J(R) is reduced, then $J(R) = 0$.
The converse is obvious. ■

Finally, there is an investigation of the Von Neumann regularity of whose simple singular right R–Modules are N–flat.

Theorem 3.3: If R is a ring satisfying condition (*) and right weakly continuous whose every simple singular right R–module is N–flat, then R is a strongly regular ring.

Proof : From Theorem 3.1 $J(R) \cap Y(R) = 0$. Since R is weakly continuous, then $J(R)= Y(R) = 0$ and R is strongly regular ring. ■

Lemma 3.4. [6]: Let R be a semi–prime ring. Then R is reduced if it is a reversible ring.■

Following [10] a right R–module M is said to be **Wjcp–injective** if for $a \notin Y(R)$, there exists a positive integer n such that $a^n \neq 0$ and every right R–homomorphism from a^nR to M can be extended to one of R to M. If R_R is Wjcp–injective, we call R is a right Wjcp–injective ring.

Before closing this section, we present the connection between simple singular N-flat and π-biregular rings.

Theorem 3.5: Let R be a semi–prime and reversible ring whose every simple singular right R–module is either Wjcp–injective or N–flat. Then R is a π-biregular ring.

Proof : For any $0 \neq a \in R$, $l(RaR)=r(RaR)=r(a)=l(a)$ by Lemma 3.4. If $Ra^nR \oplus r(a^n) \neq R$, then there exists a maximal right ideal M of R containing $Ra^nR \oplus r(a^n)$. If M is not essential in R, then $M = r(e)$, $0 \neq e^2 = e \in R$. Therefore $ea=0$. Since R is reversible, then $ae = 0$. Hence $e \in r(a) \subseteq r(e)$, which is a contradiction. So M is essential in R. By hypothesis R / M is either Wjcp–injective or N–flat. First we assume that R / M is Wjcp–injective and $a \notin Y(R)$. Hence, there exists a positive integer n such that $a^n \neq 0$ and any right R–homomorphism, $a^n R \to R/M$ can be extended to $R \to R/M$.
Set $f : a^nR \to R/M$ defined by $f(a^nr) = r + M$, $r \in R$. Then f is well-defined right R – homomorphism. Hence, there exists c □ R such that $f(a^n r) = ca^nr + M$. So $1 + M = f(a^n) = ca^n + M$, that is $1 - ca^n \in M$. Since $ca^n \in Ra^nR \subseteq M$, then $1 \in M$, which is a contradiction.
Hence R/M is N–flat. Since $a \in M$, then $a^n \neq 0$, $a^n = da^n$ for some $d \in M$ and a positive integer n. Now $1-d \in l(a^n) = r(a^n) \subseteq M$, which implies that $1 \in M$, again a contradiction. Hence $Ra^nR \oplus r(a^n) = R$, therefore R is π - biregular. ■

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