On Maximal Chains In Partially Ordered Sets With Compatible (Left, Right)-Group Actions

Abdul Aali J. Mohammad **Eman M. Tahir**

Department of Mathematics
College of Education
University of Mosul, Mosul, Iraq

Received on: 09/01/2012 **Accepted on: 19/04/2012**

ABSTRACT

In this paper, we deal with compatible (left, right)-group actions on posets, i.e; (G, H)-posets. Our main purpose in this work is to study the maximal chains in (G, H)-posets to observe that this study gives us indications on the type of some (G, H) actions on posets. Therefore, we shall study the behavior of the (G, H) actions on chains.

Keywords: maximal chains, partially ordered sets, compatible (left, right) group actions.

حول السلاسل العظمى في المجموعات المرتبة جزئيًا مع أفعال زمرتية ثنائية (أيسر، أيمن)

عبد العالي محمد **إيمان طاهر**

قسم الرياضيات، كلية التربية، جامعة الموصل

تاريخ استلام البحث: 2012/01/09 **تاريخ قبول البحث: 2012/04/19**

الملخص

في هذا البحث نتعامل مع افعال زمرتية ثنائية (ايمن , ايسر) على المجموعات المرتبة جزئياً لزمرة او زمرتين والقابلين للمبادلة . هدفنا الرئيسي في هذا البحث دراسة السلاسل العظمى في بعض المجموعات المرتبة جزئياً والتي عليها افعال زمرتية ثنائية , لاحظنا ان هذه الدراسة تعطينا بعض المؤشرات عن نوعية هذه الافعال . لذا سنحتاج إلى دراسة سلوك الافعال الزمرتية الثنائية على السلاسل.

الكلمات المفتاحية: سلاسل عظمى، مجموعات مرتبة جزئياً، أفعال زمرتية ثنائية.

§.1 Introduction:

The idea of a group action of groups on sets can be extended on sets with additional mathematical structures, specially on posets.

A group action of a group G in a poset P can be considered as a group homomorphism $\rho : G \longrightarrow isom(P , P)$ defined by $\rho(g) = \rho g$ where $\rho g : P \longrightarrow P$ is an isomorphism defined by $\rho g(p) = gp$ for all $g \in G$, $p \in P$.

Such a poset P with a left action of G on it, is called a left G-poset, or simply a G-poset.

Also, since in general there are many such homomorphism ρ, so maybe there is many group actions of the group G on a poset P (at least the trivial action). Hence, by G-poset we mean only left group action of G on P.

Equivalently, Let G be a group and P a poset , we say that P is a left G-poset if for every $g \in G$ and $p \in P$ there corresponds a unique element $gP \in P$ such that for all $p,q \in P$ and $g,g_1,g_2 \in G$;

(i) $ep=p$ (ii) $g_2(g_1p) = (g_2g_1)p$ (iii) if $p \rangle q$ then $gp \rangle gq$

When condition (iii) is neglected, P is called a left G-set.This definition is slightly different from the definition given in [6].Similarly we define a right H-poset. We can conclude that every G-poset P can be considered as a right H-poset (and conversely) which is defined by:

gp $= p^{g^{-1}}$ for all $p \in P$ and $g \in G$.

Also the concept of a group action on sets can be extended to compatible left, right actions on sets. For more details see: [5], [7] and [8].

§.2 Covering (G, H)-posets:

In this section, we give the definition of the (left, right) group actions on posets, and the covering poset of a given poset.

So, we begin with the formal definition first, before proceeding to explain the intuitive concept behind it.

Definition (2-1): [4].

A poset P is called (G,H)-poset if P is a left G-poset, a right H-poset and the two actions are compatible, that is for each $g \in G$, $h \in H$ and $p \in P$ there corresponds a unique element ${}^{g}p^{h}$ in P such that ${}^{g}p^{h} = {}^{g}(p^{h}) = ({}^{g}p)^{h}$.

Equivalently, let G and H be two groups and P a poset , we say that P is a (G , H)-poset if for every $g \in G$, $h \in H$ and $p \in P$ there corresponds a unique element ${}^{g}p^{h} \in P$ such that

1. ${}^{e}p^{e} = p$
2. P is a left G-poset with the action defined by: ${}^{g}p = {}^{g}p^{e}$ $\forall$ $p \in P$, $g \in G$.
3. P is a right H-poset with the action defined by: $p^{h} = {}^{e}p^{h}$ $\forall$ $p \in P$, $h \in H$.
4. $({}^{g}p)^{h} = {}^{g}(p^{h})$ $\forall$ $g \in G$, $h \in H$ and $p \in P$.
5. $p > q \Longrightarrow {}^{g}p^{h} > {}^{g}q^{h}$ $\forall$ $g \in G$, $h \in H$ and $p,q \in P$.

When condition (5) is neglected, P is called a (G, H)-set. For more details see: [5], [6] and[8].

Example (2-2)

Let the additive group Z acts on the set of the real numbers R by the action : ${}^{n}a = a+n$ $\forall$ $a \in R$, $n \in Z$, and the additive group Q acts on R from the right by the action $a^{n} = a-n$ $\forall$ $a \in R$, $n \in Q$, then R is a (Z , Q)-poset.

Also, there exists a G-poset P which is also a right H-poset , but it's not a (G , H)-poset , as in the following example; let $G = H = C_2 = \{e,a\}$ and $P = \{x,y,z,w,t,r\}$ is a poset with $x<y$, $z<w$, $t<r$. Then, P is a G-poset with the action defined by : ${}^{a}x = z$, ${}^{a}y = w$, ${}^{a}t = t$, ${}^{a}r = r$, ${}^{e}p = p$ $\forall$ $p \in P$, and a right H-poset with the action defined by : $x^{a} = x$, $y^{a} = y$, $z^{a} = r$, wa = t , pe = p $\forall$ $p \in P$. But, P is not (G, H)-poset, that is since $({}^{a}z)^{a} = z^{a} = t$ and ${}^{a}(z^{a}) = {}^{a}r = r$.

Remark (2-3):

Any G-poset P can be considered as (G, H)-poset with the trivial right action of H on P.

Also, from the definition above, we see that a left G-poset P is one left action of G on P. But, for (G, H)-poset P there are two compatible group actions one is from the left and the other from the right.

Definition (2-4): [2]

Let P be a poset. We say that the element a of P covers the element b of P if $a>b$ and there is no element $c \in P$ such that $a>c>b$.

Proposition (2-5):

Let P be a (G, H)-poset and $a,b \in P$ with a covers b , then ${}^{g}a^{h}$ covers ${}^{g}b^{h}$ $\forall$ $g \in G$ and $h \in H$.

Proof:

Suppose that ${}^{g}a^{h}$ does not cover ${}^{g}b^{h}$, then there exists, at least, an element $c \in P$ such that ${}^{g}a^{h} > c > {}^{g}b^{h}$ .Hence, $a > {}^{g^{-1}} c^{h^{-1}} > b$ and this is a contradiction . Therefore, ${}^{g}a^{h}$ covers ${}^{g}b^{h}$. ■

Definition (2-6): [1].

Let P be a poset. Then, the set, C (P) = $\{(a,b) : a \text{ covers } b\} \subset P \times P$, is called the covering poset of P.

Proposition (2-7): [1].

Let (P, ≥) be a poset, then $((P), \underset{C}{\geq})$ is a poset such that: for all (a,b) ,

$(a', b') \in C(P)$, $(a,b) \underset{C}{\geq} (a', b')$ *if and only if* $\{(a,b) = (a', b') \text{ or } b \geq a'\}$

Theorem (2-8):

Let P be a (G, H)-poset. Then C(P) is also a (G,H)-poset with an action defined by; ${}^{g}(a,b)^{h} = ({}^{g}a^{h}, {}^{g}b^{h})$ ∀ (a,b) ∈ C(P) , g∈G and h∈H .

Proof:

(i) ${}^{e}(a,b)^{e} = ({}^{e}a^{e}, {}^{e}b^{e}) = (a,b)$ ∀ (a,b) ∈C(P).
(ii) ${}^{g1}({}^{g2}(a,b)^{h2})^{h1} = ({}^{g1}({}^{g2}a^{h2})^{h1}, {}^{g1}({}^{g2}b^{h2})^{h1}) = ({}^{g1g2}a^{h2h1}, {}^{g1g2}b^{h2h1}) = {}^{(g1g2)}(a,b)^{(h2h1)}$
∀ (a,b) ∈C(P) (g1 , h1),(g2 , h2)∈G×H .
(iii) For all (a,b) , (a',b') ∈ C(P) , (g,h) ∈ G×H , with (a',b')≥(a,b) .
Then b'≥a . So ${}^{g}b'^{h} \geq {}^{g}a^{h}$. Since (a,b),(a',b')∈C(P) .
Then $({}^{g}a^{h}, {}^{g}b^{h})$, $({}^{g}a'^{h}, {}^{g}b'^{h})$ ∈C(P) . That is $({}^{g}a'^{h}, {}^{g}b'^{h}) \geq ({}^{g}a^{h}, {}^{g}b^{h})$. Hence ${}^{g}(a',b')^{h} \geq {}^{g}(a,b)^{h}$. Therefore, C(P) is a (G,H)-poset . ■

§3. (G, H)-Chains:

In this section, we study the (left, right) group actions on chains and when the trivial action is the only one.

Definition (3-1): [2].

A poset P is called a chain (or totally ordered set) if; for all a,b ∈P : a ≥ b or b≥a.

Equivalently, the poset P is called a chain if for every two different elements a,b of P either a > b or b > a .

From the definition above, we conclude that every element of a chain covers, at most, one element and covered at most by one element. Also, any chain has, at most, one maximal element 1 and one minimal element 0.

Remark (3-2): [2].

Any chain X of n elements is isomorphic to the set of natural numbers $\underline{n} = \{1,2,...,n\}$. that is there exists a bijection function $f : X \to \underline{n}$ *such that*:

$f(x_1) \geq f(x_2)$ *if and only if* $x_1 \geq x_2$.

Theorem (3-3):

Let $X = \{x_i\}_{i \in I}$ be a (G, H)-chain and I be a set of successive integers with … $x_{i-1} < x_i < x_{i+1} < ...$

If ${}^{g}x_i^{h} = x_j$ then, ${}^{g}x^{hi+r} = x_{j+r}$ ∀ i , j , i+r , j+r∈ I .

Proof:

$(i)\, Let\ \ i+1, j+1 \in I$. Since, X is a chain, then x_{i+1} covers x_i and by proposition (2-3), ${}^{g}x^{h}_{i+1}$ covers ${}^{g}x^{h}_{i}$.

Since, ${}^{g}x^{h}_{i} = xj$ then, x_{j+1} covers ${}^{g}x^{h}_{i}$. So ${}^{g}x^{h}_{i} = x_{j+1}$.

(ii) Now, we shall use the mathematical induction to prove that ${}^{g}x^{h}_{i+1} = x_{j+1}$ for i =1. Suppose $gx^{h}_{i+n} = x_{j+n}$ for r = n and i+n, j+n∈I. Since, X is a chain, then x_{i+n+1} covers x_{i+n} .So, ${}^{g}x^{h}_{i+n+1}$ covers ${}^{g}x^{h}_{i+n}$ .Now, from ${}^{g}x^{h}_{i+n} = {}^{g}x^{h}_{j+n}$ we have ${}^{g}x^{h}_{i+n+1} = x_{j+n+1}$. Therefore , ${}^{g}x^{h}_{i+r} = x_{j+r}\ \forall$ i , j , i+r , j+r ∈ I . ■

Lemma (3-4):

Let X be a (G, H)-chain and (g,h)∈G×H . If ${}^{g}x^{h}_{i} = x_t$ and $x_i<x^t$, then ${}^{g^{-1}}x_i^{h^{-1}}<x_i\ \forall$ x_i∈X .

Proof:

${}^{g}x_i^{h} = x_t \Rightarrow {}^{g^{-1}}({}^{g}x_i{}^{h})^{h^{-1}} = {}^{g^{-1}}x_t{}^{h^{-1}} \Rightarrow {}^{gg^{-1}}({}^{g}x_i{}^{h})^{hh^{-1}} = {}^{g^{-1}}x_t{}^{h^{-1}} \Rightarrow$ ${}^{g^{-1}}x_t{}^{h^{-1}} = x_i$.

Also, $x_i < x_t \Rightarrow {}^{g^{-1}}x_i{}^{h^{-1}} < {}^{g^{-1}}x_t{}^{h^{-1}}$. Therefore, ${}^{g^{-1}}x_i{}^{h^{-1}} < x_i$ ■

Definition (3-5):

Let P be a (G, H)-poset. For each p∈P the set:
Stab (G, H)(p) = {(g,h)∈G×H : ${}^{g}p^{h} = p$} is called the stabilizer of p.

Proposition (3-6):

Let X be a (G, H)-chain and (g,h)∈G×H with $g^{-1} = g$ and $h^{-1} = h$. Then, (g,h)∈$Stab_{(G,H)}(x_i)$ for all $x_i \in$ X.

Let ${}^{g}x_i^{h} = x_t$. Then, $x_i = {}^{g^{-1}}x_t{}^{h^{-1}}$. So, x_i = gxth . Suppose that $x_i \neq x_t$. Then, either $x_i < x_t$ or $x_t < x_i$. If $x_i < x_t$ then ${}^{g}x_i^{h} < {}^{g}x_t^{h}$ .So, $x_t < x_i$. That is a contradiction.

Similarly, we have a contradiction if $x_t < x_i$. Hence, since X is a chain, then $x_i = x_t$. So ${}^{g}x_i^{h} = x_i$.

Therefore, (g,h)∈Stab(G,H)(x_i) for all x_i∈X. ■

Theorem (3-7):

Let (X,≤) be a (G, H)-chain. Then, the (G, H) action on X is only the trivial action if X has 0 or 1.

Proof:

(i) Let 0 = x_1∈X and (g,h)∈G×H . Suppose that ${}^{g}x_1^{h} \neq$ x1, then $x_1<{}^{g}x_1^{h}$ [x_1=0] . Also, ${}^{g^{-1}}x_i{}^{h^{-1}} < x_1 = 0$. So, this is a contradiction.

So, ${}^{g}x_1^{h} = x_1$. Now, from theorem (3-3) we have ${}^{g}x_i^{h} = x_i$ for all x_i∈X and (g,h)∈G×H.

(ii) Let 1 = x_1∈X and (g,h)∈G×H . Suppose that ${}^{g}x_1^{h} \neq x_1$, then ${}^{g}x_1^{h}<x_1$[x_1=1]. Also, $x_1 < {}^{g^{-1}}x_1{}^{h^{-1}}$. So, this is a contradiction.

So, ${}^{g}x_1^{h} = x_1$. Now, from theorem (3-3) we have ${}^{g}x_i^{h} = x_i$ for all x_i ∈ X and (g,h) ∈ G×H . ■

Corollary (3-8):

Let P = {$p_1,p_2,\ldots,p_n$} be a (G,H)-chain with $p_1>p_2>\ldots>p_n$. Then, P is a trivial (G, H)-chain.

§.4 Maximal chains:

Finally, in this section, we will study the maximal chains in (G, H)-posets and we shall observe that the study of these kinds of chains gives us some indications on the type of some group actions on posets.

Definition (4-1): [2].

Let P be a poset and $X = \{x_i, x_{i+1},..., x_j\} \subseteq P$ be a chain such that $x_i < x_{i+1} < \ldots < x_j$, then X is called a maximal chain in P if and only if:
(i) There is no element as c ∈ P such that: $x_i < x_{i+1} < \ldots < c < \ldots < x_j$.
(ii) There is no element as k ∈ P such that: $k < x_i$ or $x_j < k$.

Proposition (4-2):

Let P be a (G, H)-poset and Y be a maximal chain in P. Then, ${}^gY^h$ is also a maximal chain in P with $|{}^gY^h| = |Y|$.

Proof:

(i) Since Y is a maximal chain in P , so we can say Y = {$x_i,x_{i+1},\ldots,x_j$}such that x_{r+1} covers x_r for all $i< r < j$. So ,${}^gY^h$ = {${}^gx_i^h$, ${}^gx_{i+1}^h$, ... , ${}^gx_j^h$} for all (g,h)∈G×H . Hence, ${}^gx_i^h < {}^gx_{i+1}^h < \ldots < {}^gx_j^h$ .Suppose that there exists an element as c∈P such that ${}^gx_i^h < {}^gx_{i+1}^h < \ldots < c < \ldots < {}^gx_j^h$.
Then, ${}^{g^{-1}}({}^gx_i{}^h)^{h^{-1}} < {}^{g^{-1}}({}^gx_{i+1}{}^h)^{h^{-1}} < .< {}^{g^{-1}}c^{h^{-1}} < .< {}^{g^{-1}}({}^gx_j{}^h)^{h^{-1}}$
That is $x_i < x_{i+1} < \ldots < {}^{g^{-1}}c^{h^{-1}} < \ldots < x_j$ and this is a contradiction since Y is a maximal chain.

(ii) Suppose that there exists an element b∈P such that b < gxih then:
$b \leq {}^gx_i^h \Rightarrow {}^{g^{-1}}b^{h^{-1}} < x_i \Rightarrow {}^{g^{-1}}b^{h^{-1}} = x_i \Rightarrow b = {}^gx_i^h$.
Similarly, if gxih ≤ a then gxjh = a . Therefore, gYh is a maximal chain.
Now, let the map f: Y → gYh is defined by: f(y) = gYh ∀ y∈Y.
f is injective map since : f(y1) = f(y2) ⇒ ${}^gy_1^h = {}^gy_2^h \Rightarrow y_1 = y_2$.
Also f is onto since if $x \in {}^gY^h$ then there exits y∈Y such that
$x = {}^gy^h$. Hence, f is bijection and $|Y| = |{}^gY^h|$ ■

Definition (4-3): [2].

Let P be a poset and x ∈ P. Then, the subset C of P is called a cutset of the element x in P if every element of C is not comparable with x and all the maximal chains in P cut with C∪{x}. We shall denote to this set by cut x.

Proposition (4-4):

Let P be a (G, H)-poset and C is the cutset of x ∈P. Then, ${}^gC^h$ is the cutset of gxh. That is ${}^gC^h$ = cut ${}^gx^h$.

Proof:

Let $y \in$ cut ${}^{g}x^{h}$ then ${}^{g^{-1}}y^{h^{-1}}$ is not comparable with ${}^{g}x^{h}$. So ${}^{g^{-1}}y^{h^{-1}}$ is not comparable with x. That is ${}^{g^{-1}}y^{h^{-1}} \in$ C. So, $g({}^{g^{-1}}y^{h^{-1}})h \in {}^{g}C^{h}$. That is $y \in {}^{g}C^{h}$. Hence, cut ${}^{g}x^{h} \subseteq {}^{g}C^{h}$.

Now let ${}^{g}s^{h} \in {}^{g}C^{h}$. Then, $s \in C$. So, s is not comparable with x. that is ${}^{g}s^{h}$ is not comparable with ${}^{g}x^{h}$. So ${}^{g}s^{h} \in$ cut ${}^{g}x^{h}$.

Therefore, ${}^{g}C^{h} =$ cut ${}^{g}x^{h}$. ■

Theorem (4-5):

Let P be a finite (G, H)-poset with $P(M) = \{M_1, M_2, \ldots, M_n\}$ be the set of the maximal chains in P with $|M_i| = |M_j|$ if and only if i = j. Then, the trivial action is the only action of (G, H) on P.

Proof:

To prove this theorem, we must first prove that ${}^{g}M_i^{h} = M_i$ for $1 \le i \le n$, after that we must show that ${}^{g}x^{h} = x$ for all $x \in$ Mi and $(g,h) \in G \times H$.

First part:

Our argument proceeds by mathematical induction on the number n to prove that ${}^{g}M_i^{h} = M_i$ for all $1 \le i \le n$.

Let $|M_i| = r_i$, $\forall$ $1 \le i \le n$ such that $r_1 < r_2 < \ldots < r_n$.

(i) Let n=2. That is $P(M) = \{M_1, M_2\}$ with $|M_1| \neq |M_2|$.

Suppose that ${}^{g}M1^{h} \neq M_1$. By proposition(4-2) ${}^{g}M_1^{h}$ is a maximal chain and $|{}^{g}M_1^{h}| = |M_1|$,then ${}^{g}M_1^{h} \in P(M)$. So ${}^{g}M_1^{h} = M_2$. Hence, $|{}^{g}M_1^{h}| = |M_2| = |M_1|$.That is a contradiction.So, ${}^{g}M_1^{h} = M_1$. Similarly, we have ${}^{g}M_2^{h} = M_2$.

(ii) Now assume that n=k with ${}^{g}Mi^{h} = M_i$ for all $1 \le i \le k$.

Let n=k+1. Since ${}^{g}Mi^{h} = M_i$ for all $1 \le i \le k$.

Suppose that ${}^{g}M_{k+1}^{h} \neq M_{k+1}$ then, ${}^{g}M_{k+1}^{h} = M_j$ for some $1 \le j \le k$. So, $|{}^{g}M_{k+1}^{h}| = |M_j| = r_j$. But $|{}^{g}M_{k+1}^{h}| = |M_{k+1}| = r_{k+1}$. Hence, $r_j = r_{k+1}$, that is j=k+1, and this is a contradiction since k+1>j. So, ${}^{g}M_{k+1}^{h} = M_{k+1}$.

Second part:

Since $\{M_i\}_{i=1}^{n}$ is the family of the maximal chains in P , then M_i is a finite maximal chain in P. Using corollary (3-8), we get : ${}^{g}x^{h} = x$ for all $x \in M_i$, $(g , h) \in G \times H$ with $1 \le i \le n$.

Therefore, from part one, the action of (G, H) on P is the trivial action only. ■

Definition (4-6):

Let $(H, *^{op})$ be a group. Define H^{op} to be a group its elements are the element of H and the product $h_1 *^{op} h_2 = h_2 * h_1$.

Proposition (4-7):

Let P be a (G,H)-poset , so for all $(g,h) \in G \times H$ there exists a permutation $g\rho h$: $P \to P$ defined by $g\rho h(p) = {}^{g}p^{h}$ for all $p \in P$.

Also the map $\rho : (G \times H^{op}) \to S_{|P|}$ is defined by: $\rho(g, h) = g\rho h$ for all $(g,h) \in G \times H$ is a homomorphism .

Proof:

Similar to the proof in [8] .

Hence, from every (G, H)-poset, we can get a ($G\times H^{op}$)-poset by the action $^{(g,h)}p = {}^{g}p^{h}$ $\forall$ g∈G, h∈H, p∈P. ■

Definition (4-8):

A (G, H)-poset is called injective if the corresponding homomorphism ρ is injective.

Proposition (4-9):

Let P (M) = $\{M_1, M_2, \ldots, M_n\}$ be the set of the maximal chains in the (G,H)-poset P. Let $^{g}M_i^{h} = M_i$, then, $^{g}M_j^{h} \neq M_t$ for all $j \neq i$.

Proof:

Suppose that $^{g}M_j^{h} = M_i$ for some $j \neq i$. Then, $^{g}M_j^{h} = {}^{g}M_i^{h}$ for some $j \neq i$. So $^{g^{-1}}({}^{g}M_j^{h})^{h^{-1}} = {}^{g^{-1}}({}^{g}M_i^{h})^{h^{-1}}$ for some $j \neq i$.

Hence, $M_j = M_i$ for some $j \neq i$. This is a contradiction since $j \neq i$ implies $|P(M)| < n$. Therefore, $^{g}M_j^{h} \neq M_t$ for all $j \neq i$. ■

Proposition (4-10):

Let P be an injective (G, H)-poset, and P(M) = $\{M_1, M_2, \ldots, M_n\}$ be the family of the maximal chains in P. Then:

(i) ($|M_i| = |M_j|$ if and only if i = j), implies that (G, H) = {(e,e)}.

(ii) If $|M_1| = |M_2| = \ldots = |M_n|$, then $|(G,H)| \leq n!$.

(iii) If we reordered the maximal chains such that:

$$|N_1| = |N_2| = \ldots = |N_r| \neq |N_{r+1}| = \ldots = |N_t| \neq |N_{t+1}| = \ldots = |N_n| ,$$

with $N_i \in P(M)$, $1 \leq i \leq n$, then : $|(G,H)| \leq r! \text{ x}(t-r)! \text{ x} \ldots \text{x}(n-k)!$.

Proof:

(i) Since ρ((g,h)) = ($^{g}\rho^{h}$)(p) = p = I(p) for all p∈P and (g,h)∈G×H , then (g,h)∈ker(ρ). But ker(ρ) = {e,e} because ρ is injective .

Then, (g,h) = (e,e) for all (g,h)∈G×H . So, (G, H) =ker(ρ) = {(e,e)}.

(ii) $|M_1| = |M_2| = \ldots = |M_n|$. So for all $M_i \in P(M)$ and (g,h)∈G×H there exists some $M_t \in P(M)$ such that $^{g}M_j^{h} = M_t$.From proposition (4-7) ,we have $^{g}M_j^{h} \neq M_t$ for all $j \neq i$

So, the number of permutations on the maximal chains is n!.Now, since P is an injective (G, H)-poset, then $|(G, H)| \leq n!$.

(iii) Applying (ii) on every part of equal parts of:

$$|N_1| = |N_2| = \ldots = |N_r| \neq |N_{r+1}| = \ldots = |N_t| \neq |N_{t+1}| = \ldots \neq |N_{k+1}| = \ldots = |N_n|$$

We get the number of permutations on the equal parts are, r!, (t-r)!,…,(n-k)! respectively. Using the fundamental principle of counting, the number of the permutations on the maximal chains is r! x(t-r)! x … x(n-k)! . Since, P is an injective (G, H)-poset,Then, $|(G, H)| \leq r! \text{ x}(t-r)! \text{ x} \ldots \text{x}(n-k)!$. ■

REFERENCES

[1] Behrendt. Gerhard, “Automorphism Groups of Covering Posets and of Dense Posets", Proceeding of Edinburgh Mathematical Society (1992) 35, 115-120.

[2] Birkhoff G., “Lattice Theory”, Amer. Math. Soc. Coll. Pub. Vol. XXV, Third Edition, (1967).

[3] Hanlon P., “The Incidence Algebra of a Group Reduced Partially Order Set” (1980) Combinatorial Math. 7, Springer. No.829.

[4] Mohammad A.J."On colouring maps and semidirect product groupoids on group-posets". J. Edu. & Sci., Vol. 19, No.2, (2007), 99-109.

[5] Mohammad A.J. & Khalel L.A. "Kinds of Crossed Modules",J. Edu. & Sci., Vol. 22 No. 2. , (2009) , 354-369.

[6] Mohammad A.J. & Mohammad S.A. “β-operations on Finite Posets”, J. Edu. & Sci., Vol. 19, (1994), 104-114.

[7] Morris I. & Wensley C.D., “Adams Operations and 2-operations in β-Rings”, Discrete Mathematics, 50, (1984), 253-270.

[8] Rose J.S., “ A Course on Group Theory”, Cambridge University Press, Cambridge, (1978).