Automatic Continuity of Dense Range Homomorphisms into Multiplicatively Semisimple Complete Normed Algebras

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الملخص

المسالة المفتوحة الآتية تنص على انه: إذا كان $\phi: A \rightarrow B$ تطبيق متشاكل ذا مستقر كثيف من جبر باناخ A إلى جبر باناخ B بحيث إن B شبه بسيطة. هل أن ϕ مستمرة تلقائيا؟ (انظر[1]).

في [5] أعطي حلا جزئيا للمسالة أعلاه كالآتي:

ليكن A و B جبور فريجيت بحيث إن B شبه بسيطة، نصف القطر الطيفي r_B مستمر على B و نصف القطر الطيفي r_A مستمر عند الصفر. إذا كان $\phi: A \rightarrow B$ تطبيق متشاكل ذا مستقر كثيف، عندئذ ϕ مستمرة تلقائيا.

في هذا البحث برهنا النتيجة التالية:

إذا كان $\phi: A \rightarrow B$ تطبيق متشاكل ذا مستقر كثيف من جبر معياري كامل غير تجميعي A إلى جبر معياري كامل غير تجميعي B بحيث إن B شبه بسيطة وجبر المضروبات لـ $B(M(B))$ شبه بسيطة أيضاً، نصف القطر الطيفي $\rho_{M(B)}$ هو مستمر على $M(B)$ ونصف القطر الطيفي $\rho_{M(A)}$ مستمر عند الصفر، عندئذ ϕ مستمرة تلقائيا.

ABSTRACT

The following open problem state that: If $\phi: A \rightarrow B$ is a dense range homomorphism from Banach algebra A into Banach algebra B such that B is semisimple. Is ϕ automatically continuous? (see[1])

In [5] given a partial solution of the above problem as follows:

Let A and B be a Fréchet algebras such that B is semisimple, the spectral radius r_B is continuous on B and the spectral radius r_A is

continuous at zero. If $\phi: A \to B$ is a dense range homomorphism, then ϕ is automatically continuous.

In this paper, we prove the following result:

If $\phi: A \to B$ is a dense range homomorphism from a complete normed nonassociative algebra A into a complete normed nonassociative algebra B such that B is semisimple and multiplication algebra $M(B)$ of B is also semisimple, the spectral radius $\rho_{M(B)}$ is continuous on $M(B)$ and the spectral radius $\rho_{M(A)}$ is continuous at zero, then ϕ is automatically continuous.

1. Introduction

If A and B are Banach algebras, B is semisimple and $\phi: A \to B$ is a dense range homomorphism, then the continuity of ϕ is along-standing open problem.

This is perhaps the most interesting open problem remains unsolved in automatic continuity theory of the Banach algebras.(see[1]).

We recall that from [4], the radical of an algebra A, denoted by rad A, is the intersection of all maximal left(right) ideals in A. The algebra A is called semisimple if rad$A$ ={0}. In [5], for the algebra A the spectrum of an element $x \in A$ is the set of all $\lambda \in \mathbb{C}$ such that $\lambda 1 - x$ is not invertible in A and is denoted by $Sp(x)$(or by $Sp_A(x)$). Thus $Sp(x) = \{\lambda \in \mathbb{C} : \lambda 1 - x \notin Inv(A)\}$.

Also let A be Banach algebra, then the spectral radius of x (with respect to A) is denoted by $r(x)$ (or $r_A(x)$) and is defined by the formula $r(x) = Sup\{|\lambda| : \lambda \in Sp(x)\}$.

If $(A, \|.\|)$ is a Banach algebra (not necessarily commutative) then $r_A(x) = \lim_{n \to \infty} \|x^n\|^{\frac{1}{n}} \le \|x\|$.

It is known that for any algebra A we have:

$$radA = \{x \in A : r_A(xy) = 0 \ \text{for every } y \in A\}.$$

From [9], for X, Y normed spaces and T a linear mapping from X into Y, then the separating subspace $S(T)$ of T is defined as follows:

$$S(T) = \{y \in Y : \exists \{x_n\} \subseteq X, x_n \to 0, Tx_n \to y, \text{where } n \in IN\}.$$

Proposition 1.1

Let A, B be normed algebras (complete). If $\phi: A \to B$ is a dense range homomorphism, then $S(\phi)$ is a closed ideal of B.

Proof: see[6].

We recall from [2] that, an annihilator of algebra A (denoted by $Ann(A)$) is defined as follows: $Ann(A)=\{x\in A: ax=xa=0, \forall a\in A\}$ and we say that A is zero annihilator if $Ann(A)=\{0\}$. In [7] the multiplication algebra of A denoted by $M(A)$ is defined as a subalgebra of $L(A)$ (the algebra of all linear mapping on A) generated by following operators:

$$Id_A : A\to A \qquad ,L_x : A\to A \qquad ,R_x : A\to A$$
$$a\mapsto Id_A(a)=a \qquad a\mapsto L_x(a)=xa \qquad a\mapsto R_x(a)=ax$$

Where $a,x\in A,$ which are called identity, left and right multiplication operators respectively.

Proposition 1.2 [7]

Let A,B be normed algebras, $\phi : A\to B$ is a dense range homomorphism. Then $\hat{\phi} : M(A)\to M(B)$ is a dense range homomorphism given by the relation :

$$\phi F=\hat{\phi}(F)\phi \qquad \forall F\in M(A) \quad ....(1).$$

Proposition 1.3

If ϕ is a dense range homomorphism from a normed algebra A into a normed algebra B, then

1. $S(\hat{\phi})(B)\subseteq S(\phi).$
2. $L_{S(\phi)}\cup R_{S(\phi)}\subseteq S(\hat{\phi})$, where $L_{S(\phi)}=\{L_x : x\in S(\phi)\}$, $R_{S(\phi)}=\{R_x : x\in S(\phi)\}$.

Proof:

1. To prove that $S(\hat{\phi})(B)\subseteq S(\phi)$, we first prove that $S(\hat{\phi})(\phi(A))\subseteq S(\phi).$ Let $a\in A$, let $T\in S(\hat{\phi})$ and $\{F_n\}$ be a sequence of continuous operators in $M(A)$, such that $\{F_n\}\to 0$ and $\{\hat{\phi}(F_n)\}\to T.$

   From strange operator topology (SOT), we obtain $\{F_n(a)\}\to 0$ and
   $\{\phi(F_n(a))\}=\{(\phi F_n)(a)\}=\{(\hat{\phi}(F_n)\phi)(a)\}=\{(\hat{\phi}(F_n)(\phi(a))\}\to T(\phi(a))$.
   Therefore, $T(\phi(a))\in S(\phi)$, for all $T\in S(\hat{\phi})$, $a\in A$. i.e. $S(\hat{\phi})(\phi(A))\subseteq S(\phi)$. Note that,

   $$S(\hat{\phi})(B)=S(\hat{\phi})(\overline{\phi(A)})$$
   $$\subseteq \overline{S(\hat{\phi})}(\overline{\phi(A)})$$
   $$\subseteq \overline{S(\phi)}=S(\phi) \qquad \text{(by proposition(1.1)).}$$

2. Let $b \in S(\phi)$. Then $\exists \{a_n\} \subseteq A$ such that $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} \phi(a_n) = b$. Therefore, $\lim_{n\to\infty} L_{a_n} = 0$ and $\lim_{n\to\infty} L_{\phi(a_n)} = L_b$. This implies that $L_b \in S(\hat{\phi})$. Similarly, we can proof that $R_b \in S(\hat{\phi})$.

2. Fundamental Results

In this section we prove our fundamental following results:

Theorem 2.1

Let $\phi : A \to B$ be a homomorphism with dense range from normed algebra A into normed algebra B then $S(\hat{\phi})$ is a closed ideal of $M(B)$.

Proof:

Clearly $S(\hat{\phi})$ is a closed linear subspace of $M(B)$. Let $G \in S(\hat{\phi})$ and $Z \in \hat{\phi}(M(A))$. There exists a sequence $\{F_n\}$ in $M(A)$ such that $\{F_n\} \to 0$ and $\{\hat{\phi}(F_n)\}) \to G$. Note that, $Z = \hat{\phi}(F)$ for some $F \in M(A)$. Hence,

$\{FF_n\} \to 0$ *and* $\{\hat{\phi}(FF_n)\} = \hat{\phi}(F)\hat{\phi}(\{F_n\}) \to ZG \in S(\hat{\phi})$. similarly, $GZ \in S(\hat{\phi})$. Therefore, $S(\hat{\phi})$ is an ideal of $\hat{\phi}(M(A))$. Hence, $\hat{\phi}(M(A))S(\hat{\phi}),\ S(\hat{\phi})\hat{\phi}(M(A)) \subseteq S(\hat{\phi})$ and this implies

$$\overline{\hat{\phi}(M(A))}\,\overline{S(\hat{\phi})} \subseteq \overline{S(\hat{\phi})} \text{ and } \overline{S(\hat{\phi})}\,\overline{\hat{\phi}(M(A))} \subseteq \overline{S(\hat{\phi})}.$$

Thus $M(B)S(\hat{\phi}) \subseteq S(\hat{\phi})$ and $S(\hat{\phi})M(B) \subseteq S(\hat{\phi})$ as required.■

Theorem 2.2

Let $\phi : A \to B$ be a dense range homomorphism from complete normed nonassociative algebra A into complete normed nonassociative algebra B such that B is semisimple and $M(B)$ is also semisimple, the spectral radius $\rho_{M(B)}$ is continuous on $M(B)$ and the spectral radius $\rho_{M(A)}$ is continuous at zero, then ϕ is automatically continuous.

Proof:

According to the proposition (1.2) there exists homomorphism with dense range $\hat{\phi} : M(A) \to M(B)$ given by the relation $\phi F = \hat{\phi}(F)\phi$.

For every $G \in S(\hat{\phi})$ There exists a sequence $\{F_n\} \subseteq M(A)$ such that $\{F_n\} \to 0$ in $M(A)$ and $\hat{\phi}(\{F_n\}) \to G$ in $M(B)$. Since $\rho_{M(A)}$ is continuous at zero by assumption, we have $\rho_{M(A)}(F_n) \to 0$, then $\rho_{M(B)}(\hat{\phi}(F_n)) \to 0$.

On the other hand, again by continuity of $\rho_{M(B)}$ we have $\rho_{M(B)}(\hat{\phi}(F_n)) \to \rho_{M(B)}(G)$. Hence,

$$\rho_{M(B)}(G) = 0 \ldots\ldots\ldots\ldots\ldots(2)$$

Since $\hat{\phi}: M(A) \to M(B)$ is a dense range homomorphism by theorem(2.1) $S(\hat{\phi})$ is an ideal in $M(B)$. Thus for every $Z \in M(B)$, $GZ \in S(\hat{\phi})$. By (2) we get $\rho_{M(B)}(GZ) = 0$. Since $M(B)$ is semisimple, we have:

$$rad\, M(B) = \{G \in M(B) : \rho_{M(B)}(GZ) = 0 \text{ for every } Z \in M(B)\} = \{0\}.$$

Therefore, $G \in rad\, M(B)$. So $S(\hat{\phi}) \subseteq rad\, M(B)$. Hence, we have $S(\hat{\phi}) = \{0\}$ and according the proposition (1.3)(2) we get $L_{S(\phi)} \cup R_{S(\phi)} \subseteq S(\hat{\phi})$ and this imply $L_{S(\phi)} = R_{S(\phi)} = 0$. Thus, $S(\phi) \subseteq Ann(B)$ and since $Ann(B) = 0$ then $S(\phi) = 0$. By closed graph theorem we get ϕ continuous.■

3. An application example

We recall from [8] that , the intersection of full subalgebras of an associative algebra A is another full subalgebra of A it follows that for any nonempty subset S of A there is a smallest full subalgebra of A which contains S. This subalgebra will be called the full subalgebra of A generated by S.

Now let A be a nonassociative algebra. The full subalgebra of $L(A)$ generated by $L_A \cup R_A$ will be called the full multiplication algebra of A and will be denoted by $FM(A)$.

Consider the set $W(A)$ of those elements a in A for which L_a and R_a belong to the Jacobson radical of $FM(A)$, $W(A)$ is a subspace of A so it contains a largest subspace invariant under the algebra of operators $FM(A)$. This last subspace, which is clearly a two-sided ideal of A, will be called the weak radical of A and denoted by **w-Rad(*A*)**.

Let A be nonassociative algebra and let C be any subalgebra of $L(A)$ such that $L_A \cup R_A \subset C \subset FM(A)$. As in the definition of weak radical we can consider the largest C-invariant subspace of A consisting of elements a such that L_a and R_a lie in the Jacobson radical of C. This subspace will be called the C-radical of A and denoted by $C - Rad(A)$. The ultra-weak radical of A (uw-***Rad(A)***) is defined as the sum of all the C-radicals of A when C runs through the set of all subalgebras of $L(A)$ satisfying $L_A \cup R_A \subset C \subset FM(A)$.

Proposition 3.1

Let ϕ be a homomorphism from a complete normed nonassociative algebra A into a complete normed nonassociative algebra B. Assume that the ultra-weak radical of B is zero. Then T is continuous.

Proof: (see[3],[8]).

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