Numerical solutions of Fredholm integral equation of the first kind with degenerated kernel by using Hermite polynomial

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الملخص:

في هذا البحث استخدمنا متعددات حدود هيرمت مع طرق كالكن ،شملت التوضيح و المربعات الصغرى لحل معادلات فريدهولم التكاملية من النوع الاول ذات نـواة المنحلـة،وتم اقتـراح خوارزمية لكل من هذه الطرق, واختبرت الخوارزمية على عدد من الامثلة العددية، و كانت النتائج مقبولة عند مقارنتها مع الحل المضبوط، كما يتبين في جداول(4,3,2,1)، استخدمت Matlab لكتابة جميع البرامج الخاصة بالموضوع.

Abstract:

In this paper we used Hermite polynomial with Galerkin , collocation and least square methods for solving fredholm integral equations of the first kind (F.I.E.F.K.) with degenerated kernel .The algorithms and computer applications of the algorithms are given for the methods which have tested through numerical examples and compared the results with exact solutions as indecated in tables (1,2,3,4),(used Matlab programming in this work).

Key words: Hermite polynomial, Galerkin, collocation, least square method, matlab program.

1-Introduction.

An equation, which contains an unknown function under the integral symbol, is called integral equation. As general example for integral equation may be expressed as fallows:-

$$u(x) = g(x) + \int k(x,t)u(t)dt \qquad (1)$$

Where $g(x)$ and $k(x,t)$ are known functions, $u(x)$ is the unknown function. The function $k(x,t)$ is the kernel of integral equation ;(the function $g(x)$ is known as a Driving term); for more details see [2, 5, 10, 11].

To study the standard methods for integral equation of the form

$$h(x)u(x) = g(x) + \lambda \int_a^b k(x,t)u(t)dt \qquad (2)$$

Where x and t are real variables varying in the interval $[a,b]$,and λ is numerical factor, when $h(x) = 0$ we will obtain

$$g(x) = \lambda \int_a^b k(x,t)u(t)dt \qquad (3)$$

This is called (F.I.E.F.K.) .There is several Methods for solving (F.I.E.F.K.) like Galerkin, collocation [1]...etc.

Definition. (Degenerated kernel).

Suppose that $k(x,t)$ is a kernel defined on the square $[a,b]*[a,b]$ and there are a finitely many functions $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ such that

$$k(x,t) = \sum_{i=1}^{n} a_i(x)b_i(t) \qquad (a \le x, t \le b)$$

In this case the kernel is said to be degenerated kernel [1, 4].

Definition. (Orthogonal function).

A set of function $\{\phi_i(x)\}, i = 0,1,\dots$ is said to be orthogonal for the interval $[a,b]$ with respect to weight function $w(x) \ge 0$ if

$$\int_a^b w(x)\phi_i(x)\phi_j(x)dx = 0 \qquad \text{, for } i \ne j$$

$$\int_a^b w(x)\phi_i(x)\phi_j(x)dx = \alpha_k > 0 \quad \text{, for } i = j$$

If, in addition $\alpha_k = 1$ for each $k = 0,1,\dots,n$ the set is said to be orthonormal.

A special case of orthogonal function consists of the set of orthogonal polynomial $\{\phi_n(x)\}$,where n denotes the degree of the polynomials $\phi_n(x)$.

3-Hermite polynomials (H.P.).

The Hermite polynomials are defined through

$$H_n(x) = (-1)^n \exp(x^2)\frac{d^n}{dx^n}\exp(-x^2) \quad ,n = 0,1,2,..$$

and they satisfy the differential equation

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$$

The Hermite polynomials are a set of orthogonal polynomials over the interval $(-\infty,\infty)$,with the weight function $w(x) = \exp(-x^2)$.

The recurrence relation of them is [3].

$$\begin{aligned} &H_0(x) = 1, H_1(x) = 2x \\ &H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \end{aligned} \quad ,n \ge 1 \qquad (4)$$

3- Hermite polynomial to obtain error function, for (F.I.E.F.K.).

In this paper the orthogonal (H.P.) accompanied with the Galerkin, collocation and least square methods will be used to find the approximate solution of **(F.I.E.F.K.).**

An approximate solution of the form

$$f_n(x) = \sum_{j=0}^{n} \alpha_j H_j(x) \qquad (5)$$

will be assumed, where $H_j(x)$ are the (H.P.) defined in (4).Recall equation (3) .Which is **(F.I.E.F.K.)** to find approximate solution of equation (3) substituting $f_n(x)$ in equation (3) gives

$$g(x) = \int_a^b k(x,t) f_n(t)dt + E_n(x,\alpha_0,\alpha_1,...,\alpha_n) \qquad (6)$$

This leads to

$$g(x) = \int_a^b k(x,t)\sum_{j=0}^{n}\alpha_j H_j(t)dt + E_n(x,\alpha_0,\alpha_1,...,\alpha_n) \qquad (7)$$

$$\Rightarrow E_n(x,\alpha_0,\alpha_1,...,\alpha_n) = g(x) - \sum_{j=0}^{n}\alpha_j \int_a^b k(x,t)H_j(t)dt \qquad (8)$$

where $E_n(x,\alpha_0,\alpha_1,...,\alpha_n)$ is the error.

Let $\Psi_j(x) = \int_a^b k(x,t)H_j(t)dt \quad , j = 0,1,...n \qquad (9)$

Substuting in equation (8) this leads to

$$\Rightarrow E_n(x,\alpha_0,\alpha_1,...,\alpha_n) = g(x) - \sum_{j=0}^{n}\Psi_j(x)\alpha_j \qquad (10)$$

4-Using Hermite polynomial with Galerkin (H.P.G.) method to solve (F.I.E.F.K.).

In this method, the error function $E_n(x,\alpha_0,\alpha_1,...,\alpha_n)$ is made orthogonal to m linearly independent functions $H_0,H_1,...,H_m$ on the interval $[a,b]$ that is [1, 8]

$$\int_a^b H_i(x)w(x)E_n(x,\alpha_0,\alpha_1,...,\alpha_n)dx = 0 \quad ,i = 0,1,...,m \qquad (11)$$

This leads to

$$\int_a^b H_i(x)w(x)[g(x) - \sum_{j=0}^{n} \Psi_j(x)\alpha_j]dx = 0 \qquad (12)$$

This leads to :

$$-\sum_{j=0}^{n}[\int_a^b H_i(x)w(x)\Psi_j(x)dx]\alpha_j + \int_a^b H_i(x)w(x)g(x)dx = 0 \qquad (13)$$

$$\Rightarrow \sum_{j=0}^{n}[\int_a^b H_i(x)w(x)\Psi_j(x)dx]\alpha_j = \int_a^b H_i(x)w(x)g(x)dx \qquad (14)$$

where $i = 0,1,...,m$

Let $s_{ij} = \int_{-\infty}^{\infty} H_i(x)\Psi_j(x)w(x)dx$ and

$$w_i = \int_{-\infty}^{\infty} H_i(x)g(x)w(x)dx \quad ;i = 0,1,...,m \ , j = 0,1,...,n \qquad (15)$$

This leads to the system:

$$\sum_{j=0}^{n} s_{ij}\alpha_j = w_i \quad ;i = 0,1,...,m \qquad (16)$$

Finally assume that $m = n$,this produce a system of $(n+1)$ linear equation with $(n+1)$ unknown coefficients $\{\alpha_n\}$.By solving this system we find the values of α_j's and then substituting them in equation (5) gives the solution of (3). The Simpson's 1/3 rule is used to calculate the required integrals.

Algorithm.

The following algorithm is to obtain the approximate solution by previous method.

Algorithm for (H.P.G.) method.

Step 1: Input a,b,n where $[a,b]$ bounded interval and n the number of divided interval.

Step 2: Compute $x_i = a + ih$, $h = \frac{b-a}{n}$, $i = 0,1,...,n$.

Step 3: Evaluate $\Psi_j(x), j = 0,1,...,n$ from equation (9).

Step 4: Evaluate s_{ij} and w_i form equation (15) for $i = 0,1,...,n$, $j = 0,1,...,n$.

Step 5: Formulate a system which is represented in (16).

Step 6: Solve the system (16) for coefficients α_j's.

Step 7: Finally substituting α_j's in equation (5) to obtain the approximate solution of equation (3).

5- Using Hermite polynomial with Collocation (H.P.C.) method to solve (F.I.E.F.K.).

This method presents the n conditions by insisting that the error in equation (10) vanishes at $(n+1)$ points $x_0, x_1, \ldots, x_n$ this reduces (10) at $(n+1)$ equation, i.e. at these points the error function $E_n(x, \alpha_0, \alpha_1, \ldots, \alpha_n)$ vanishes [1, 9]. That is

$$\sum_{j=0}^{n} \Psi_j(x_i)\alpha_j = g(x_i) \ ; i = 0,1,\ldots,m \qquad (17)$$

where $\Psi_j(x) = \int_a^b k(x,t)H_j(t)dt$ and

$$\Psi_j(x_i) = \int_a^b k(x_i,t)H_j(t)dt \qquad (18)$$

Let A be a matrix $A = \Psi_j(x_i)$ $i = 0,1,\ldots,m$, $j = 0,1,\ldots,n$,

$B = A'$, $z_{ij} = B$ and $v_i = g(x_i)$, $i = 0,1,\ldots,m$ (19)

This leads to

$$\sum_{j=0}^{n} z_{ij}\alpha_j = v_i \ ; i = 0,1,\ldots,m \qquad (20)$$

Finally assume that $m = n$,this produce a system of $(n+1)$ linear equation with $(n+1)$ unknown coefficients $\{\alpha_n\}$. By solving this system we find the values of α_j 's and then substituting them in equation (5), the solution of (3) is given. The Simpson's 1/3 rule is used to calculate the required integrals.

Algorithm for (H.P.C.) method.

Step 1: Input a,b,n where $[a,b]$ bounded interval and n the number of divided interval.

Step 2: Compute $x_i = a + ih$, $h = \frac{b-a}{n}$, $i = 0,1,\ldots,n$.

Step 3: For all $j = 0,1,\ldots,n$ compute $\Psi_j(x_i)$, $i = 0,1,\ldots n$ from equation (18), put

$A = \Psi_j(x_i)$, $B = A'$, $z_{ij} = B$ and $v_i = g(x_i)$ for $i = 0,1,\ldots,n$, from equation (19).

Step 4: Formulate a system which is represented in (20).

Step 5: Solve the system (20) for coefficients α_j 's.

Step 6: finally substituting α_j 's in equation (5) to obtain the approximate solution of equation (3).

6- Using Hermite polynomial with least square (H.P.L.S.) method to solve (F.I.E.F.K.).

This method is in short insists on the integral of the square of the error,

$$\int_a^b E_n^2(x,\alpha_0,\alpha_1,...,\alpha_n)dx = \min imum$$

on the interval $[a,b]$ being minimum .The necessary conditions for E_n to be minimum are

$$\frac{\partial E_n}{\partial \alpha_i} = 0 \quad \text{For each } i = 0,1,..n \qquad (21)$$

If we square (10) and perform the required partial differentiation under assumption that all the integrals involved converge the condition (21), we obtain the following system

$$\sum_{j=0}^{n} s_{ij}\alpha_j = w_i \quad , \quad i = 0,1,...,n \qquad (22)$$

where $w_i = \int_{-\infty}^{\infty} s_i(x)g(x)w(x)dx \quad , \quad i = 0,1,..m \qquad (23)$

$$s_{ij} = \int_{-\infty}^{\infty} s_i(x)s_j(x)w(x)dx \quad , i = 0,1,..m \; , j = 0,1,...n \qquad (24)$$

while $\qquad s_j(x) = \int_a^b k(x,t)H_j(t)dt \qquad (25)$

Finally assume that $m = n$,this produce a system of $(n+1)$ linear equation with $(n+1)$ unknown coefficients $\{\alpha_n\}$. By solving this system we find the values of α_j 's and then substituting them in equation (5), the solution of (3) is given. The Simpson's 1/3 rule is used to calculate the required integrals.

Algorithm for (H.P.L.S.) method.

Step 1: Input a,b,n where $[a,b]$ bounded interval and n the number of divided interval.

Step 2: Compute $x_i = a + ih \, , h = \dfrac{b-a}{n} \, , i = 0,1,...,n$.

Step 3: Evaluate $s_j(x), j = 0,1,...,n$ from equation (25).

Step 4: Evaluate s_{ij} and w_i form equation (23) and (24) respectively for $i = 0,1,...,m \, , j = 0,1,...,n$.

Step 5: Formulate a system which is represented in (22).

Step 6: Solve the system (22) for coefficients α_j 's.

Step 7: finally substituting α_j 's in equation (5) to obtain the approximate solution of equation (3).

7-Numerical examples:-

Example (1):

The problem is given by

$$\frac{x^2}{2}+\frac{2x}{3}+\frac{1}{4}=\int_0^1 (x^2+2xt+t^2)u(t)dt$$

With the exact solution $u(x)=x$

Assume the approximate solution is of the form

$$f_n(x)=\sum_{j=0}^{n}\alpha_j H_j(x)$$

and the value of h is $1/n$ Simpson's 1/3 rule had been used to solve the required integrals numerically. Results and running time are all listed in table (1) using different value of n.Table (2) shows error of the methods.

Table (1) shows a comparison between the exact solution and the approximate results (numerical solution).Obtianed by the methods.

x	n	H.P.G.	H.P.C.	H.P.L.S.	Exact
0.00	4	-0.0009	-0.0640	0.0017	0.00
0.1		0.0999	0.0869	0.1002	0.1
0.2		0.2003	0.2204	0.1994	0.2
0.3		0.3005	0.3346	0.2991	0.3
0.4		0.4004	0.4306	0.3993	0.4
0.5		0.5001	0.5127	0.4998	0.5
0.6		0.5998	0.5880	0.6004	0.6
0.7		0.6995	0.6669	0.7008	0.7
0.8		0.7995	0.7625	0.8009	0.8
0.9		0.8999	0.8913	0.9001	0.9
1.00		1.0010	1.0725	0.9982	1.00
R.T.		29.485 seconds	0.8430 seconds	33.563 seconds	
0.00	8	0.0000	-0.0084	0.0016	0.00
0.1		0.1000	0.0999	0.0998	0.1
0.2		0.2000	0.2036	0.2006	0.2
0.3		0.3000	0.3034	0.3008	0.3
0.4		0.4000	0.4012	0.4005	0.4
0.5		0.5000	0.4985	0.5000	0.5
0.6		0.6000	0.5969	0.5995	0.6
0.7		0.7000	0.6973	0.6992	0.7
0.8		0.8000	0.7996	0.7994	0.8
0.9		0.9000	0.9021	0.9002	0.9
1.00		1.0000	1.0013	1.0017	1.00
R.T.		1496.641 seconds	20.30 seconds	1631.938 seconds	

Table (2).Errors obtained in several methods.

x	n	Error H.P.G.	Error H.P.C.	Error H.P.L.S.
0.00	4	0.0009	0.0640	0.0017
0.1		0.0001	0.0131	0.0002
0.2		0.0003	0.0204	0.0006
0.3		0.0005	0.0346	0.0009
0.4		0.0004	0.0306	0.0007
0.5		0.0001	0.0127	0.0002
0.6		0.0002	0.0120	0.0004
0.7		0.0005	0.0331	0.0008
0.8		0.0005	0.0375	0.0009
0.9		0.0001	0.0087	0.0001
1.00		0.0010	0.0725	0.0018
0.00	8	0.0000	0.0084	0.0016
0.1		0.0000	0.0001	0.0002
0.2		0.0000	0.0034	0.0006
0.3		0.0000	0.0034	0.0008
0.4		0.0000	0.0012	0.0005
0.5		0.0000	0.0015	0.0000
0.6		0.0000	0.0031	0.0005
0.7		0.0000	0.0027	0.0008
0.8		0.0000	0.0004	0.0006
0.9		0.0000	0.0021	0.0002
1.00		0.0000	0.0013	0.0017

Example (2):-

The problem is given by

$$\int_0^1 k(x,t)u(t)dt = g(x) \quad ,0 \le x \le 1$$

where

$$k(x,t) = (t^2 - 2xt + x^2) \quad ,g(x) = \frac{x^2}{4} - \frac{x}{3} + \frac{1}{8}$$

With the exact solution $u(x) = \frac{x}{2}$.

Assume the approximate solution is of the form

$$f_n(x) = \sum_{j=0}^{n} \alpha_j H_j(x) \ ,$$

and the value of h is $1/n$ Simpson (1/3) rule had been used to solve the required integrals numerically. Results and running time are all listed in table (3) using different value of n. Table (4) shows error of the methods.

Table (3) shows a comparison between the exact solution and the approximate results (numerical solution) from the methods.

x	n	H.P.G.	H.P.C.	H.P.L.S.	Exact
0.0000	4	-0.0324	-0.1509	0.0300	0.0000
0.1000		0.0439	0.0221	0.0538	0.0500
0.2000		0.1106	0.1499	0.0890	0.1000
0.3000		0.1674	0.2307	0.1344	0.1500
0.4000		0.2150	0.1693	0.1879	0.2000
0.5000		0.2558	0.2765	0.2467	0.2500
0.6000		0.2936	0.2696	0.3073	0.3000
0.7000		0.3333	0.2723	0.3652	0.3500
0.8000		0.3815	0.3144	0.4154	0.4000
0.9000		0.4461	0.4323	0.4518	0.4500
1.0000		0.5363	0.6685	0.4678	0.5000
R.T.		29.61 seconds	1.547 seconds	32.516 seconds	
0.0000	6	0.0062	-0.1263	0.0340	0.0000
0.1000		0.0533	0.0172	0.0552	0.0500
0.2000		0.0988	0.1387	0.0884	0.1000
0.3000		0.1451	0.2249	0.1327	0.1500
0.4000		0.1940	0.2706	0.1862	0.2000
0.5000		0.2462	0.2805	0.2462	0.2500
0.6000		0.3009	0.2697	0.3087	0.3000
0.7000		0.3561	0.2651	0.3685	0.3500
0.8000		0.4083	0.3051	0.4186	0.4000
0.9000		0.4520	0.4404	0.4503	0.4500
1.0000		0.4803	0.7340	0.4524	0.5000
R.T.		250.50 seconds	1.547 seconds	32.516 seconds	

Table (4).Errors obtained in several methods.

x	n	Error H.P.G.	Error H.P.C.	Error H.P.L.S.
0.0000	4	0.0324	0.1509	0.0300
0.1000		0.0061	0.0279	0.0038
0.2000		0.0106	0.0499	0.0110
0.3000		0.0174	0.0807	0.0156
0.4000		0.0150	0.0693	0.0121
0.5000		0.0058	0.0265	0.0033
0.6000		0.0064	0.0304	0.0073
0.7000		0.0167	0.0777	0.0152
0.8000		0.0185	0.0856	0.0154
0.9000		0.0039	0.0177	0.0018
1.0000		0.0363	0.1685	0.0322
0.0000	6	0.0062	0.1263	0.0340
0.1000		0.0033	0.0328	0.0052
0.2000		0.0012	0.0387	0.0116
0.3000		0.0049	0.0749	0.0173
0.4000		0.0060	0.0706	0.0138
0.5000		0.0038	0.0305	0.0038
0.6000		0.0009	0.0303	0.0087
0.7000		0.0061	0.0849	0.0185
0.8000		0.0080	0.0949	0.0186
0.9000		0.0020	0.0096	0.0003
1.0000		0.0197	0.2340	0.0476

8-Conclusion.

The Hermite polynomial was used with Galerkin , collocation and least square methods to find the approximate solution of (F.I.E.F.K.).The results which obtained by using Hermite polynomial with Galerkin are better than the results which are obtained by using Hermite polynomial with collocation and least square methods, as indicated in tables (1,3), and also errors reduced, indicated in tables (2,4).

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