Investigation of solvability conditions of certain boundary value problem by perturbation method

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الملخص:

تم في هذا البحث دراسة شروط الحل لمعادلة تفاضلية جزئية معينة ذات شروط حدودية متجانسة تحتوي معلمة صغيرة. هذه المسألة حلت باستخدام أسلوب التشويش.

هنا تعاملنا مع المعادلة التفاضلية الجزئية

$$\varphi_{xx} + \varphi_{yy} + w^2\varphi = 0$$

مع الشروط الحدودية

$$\varphi_y(x,a) - \varphi_y(x,b) = 0$$

و

$$\varphi(x,y)\big|_{y=a+\varepsilon \sin k_w x} - \varphi(x,y)\big|_{y=b+\varepsilon \sin k_w x} = 0$$

حيث k_w, w ثوابت و ε معلمة صغيرة. طبقا لهذه الطريقة فان الحل للمسألة يمثل بالحدود القليلة الأولى لتبسيط التشويش.

Abstract

In this paper we study the solvability conditions under which certain partial differential equation with homogenous boundary conditions containing small parameter has a solution by using perturbation technique.

Here we deal with the following partial differential equation

$$\varphi_{xx} + \varphi_{yy} + w^2\varphi = 0$$

with boundary conditions

$$\varphi_y(x,a) - \varphi_y(x,b) = 0$$

and

$$\varphi(x,y)\big|_{y=a+\varepsilon\sin k_w x} - \varphi(x,y)\big|_{y=b+\varepsilon\sin k_w x} = 0$$

where k_w, w are constants and ε is small parameter.

According to this method, the solution of the problem is represented by the first few terms of a perturbation expansion.

1. Introduction

Many of the problems faced to day by physicists, engineers and applied mathematicians involve difficulties, such as nonlinear boundary conditions at complex known or unknown boundaries can be solved by approximation methods. One of them is perturbation method, according to these techniques, the solution of the problem is represented by the first few terms of a perturbation expansion [5], perturbation methods have been used [2] for solving elliptic equations with small nonlinearity. In [4] the author used the perturbation method which is given in [6] to find the solvability conditions for certain eigenvalue problem of fourth order. In [3] various perturbation problem are arise in the theory of lubrication.

Homotopy perturbation method is applied for solving fourth order boundary value problems by [7].

In this study we give a generalization of the boundary conditions that were given by [1]. Here we deal with the following partial differential equation

$$\varphi_{xx} + \varphi_{yy} + w^2\varphi = 0 \tag{1.1}$$

with boundary conditions

$$\varphi_y(x,a) - \varphi_y(x,b) = 0 \tag{1.2}$$

$$\varphi(x,y)\big|_{y=a+\varepsilon\sin k_w x} - \varphi(x,y)\big|_{y=b+\varepsilon\sin k_w x} = 0 \tag{1.3}$$

where k_w, w are constants and ε is small parameter.

2. Solvability conditions for partial differential equations with boundary conditions

The study is divided as the following steps:

I - Transfer the boundary conditions

Transfer the boundary conditions from

$y = a + \varepsilon\sin k_w x,\ y = b + \varepsilon\sin k_w x$ *to* $y = a\ ,\ y = b$ respectively

Let $\varphi = \varphi_0(x,y) + \varepsilon\varphi_1(x,y) + \varepsilon^2\varphi_2(x,y) + ...$ (2.1)

We note that the boundary condition (1.3) is imposed at $y = a + \varepsilon\sin k_w x,\ y = b + \varepsilon\sin k_w x$, and hence ε appears in the argument of φ as well as in the coefficients.

Since the usual procedure in perturbation methods is to equate coefficients of equal powers of ε, we will not be able to do that unless ε

is removable from the argument. To do this, we transfer the boundary conditions from

$y=a+\varepsilon\sin k_w x \quad to \quad y=a \;,\; y=b+\varepsilon\sin k_w x \quad to \quad y=b$

by a Taylor series expansion.

We write $\varphi(x,y) \;\; at \;\; y=a+\varepsilon\sin k_w x,\; y=b+\varepsilon\sin k_w x$

and expanding in a Taylor series about $y=a, y=b$, we have

$$\varphi(x,a+\varepsilon\sin k_w x)=\varphi(x,a)+\varphi_y(x,a)\varepsilon\sin k_w x+\frac{1}{2!}\varphi_{yy}(x,a)\varepsilon^2\sin^2 k_w x+......$$

$$\varphi(x,b+\varepsilon\sin k_w x)=\varphi(x,b)+\varphi_y(x,b)\varepsilon\sin k_w x+\frac{1}{2!}\varphi_{yy}(x,b)\varepsilon^2\sin^2 k_w x+......$$

Substituting these Taylor series expansions into (1.3), we obtain

$$\varphi(x,a)+\varphi_y(x,a)\varepsilon\sin k_w x+\frac{1}{2!}\varphi_{yy}(x,a)\varepsilon^2\sin^2 k_w x+.....-$$

$$-\varphi(x,b)-\varphi_y(x,b)\varepsilon\sin k_w x-\frac{1}{2!}\varphi_{yy}(x,b)\varepsilon^2\sin^2 k_w x-...... \qquad (2.2)$$

Now substituting (2.1) into (1.1,1.2 and 2.2) and equating coefficients of like powers of ε, we have

order ε^0

$$\varphi_{0xx}+\varphi_{0yy}+w^2\varphi_0=0 \qquad (2.3)$$

$$\varphi_{0y}(x,a)-\varphi_{0y}(x,b)=0 \qquad (2.4)$$

$$\varphi_0(x,a)-\varphi_0(x,b)=0 \qquad (2.5)$$

order ε^1

$$\varphi_{1xx}+\varphi_{1yy}+w^2\varphi_1=0 \qquad (2.6)$$

$$\varphi_{1y}(x,a)-\varphi_{1y}(x,b)=0 \qquad (2.7)$$

$$\varphi_1(x,a)-\varphi_1(x,b)=-\varphi_{0y}(x,a)\sin k_w x+\varphi_{0y}(x,b)\sin k_w x \qquad (2.8)$$

II - Applying method of separation of variables

Since the problem (2.3-2.5) is homogenous and with constant coefficients, therefore can be solved by separation of variables as the following

Let $\varphi_0(x,y)=X(x)Y(y)$ (2.9)

$$X''Y+XY''+w^2XY=0 \qquad (2.10)$$

$$\varphi_{0y}(x,a)-\varphi_{0y}(x,b)=0 \quad\Rightarrow\quad Y'(a)-Y'(b)=0 \qquad (2.11)$$

$$\varphi_0(x,a)-\varphi_0(x,b)=0 \quad\Rightarrow\quad Y(a)-Y(b)=0 \qquad (2.12)$$

Dividing (2.10) by XY and equating to the separation constants, we can write

$$\frac{-X''}{X}=\frac{Y''}{Y}+w^2=c \quad ,c>0$$

or

$$X''+k^2X=0 \qquad ,c=k^2 \tag{2.13}$$

$$Y''+(w^2-k^2)Y=0 \tag{2.14}$$

Both equations (2.13, 2.14) have solution

$$X = \exp\ (\mp ikx\ ) \tag{2.15}$$

$$Y=c_1\sin\sqrt{w^2-k}\,y+c_2\cos\sqrt{w^2-k}\,y \quad ,(c_1,c_2 \ \ arbitrary) \tag{2.16}$$

Substituting (2.16) in (2.11, 2.12),and we find the determinant of the coefficients and equating it to zero, we have

$$\cos\sqrt{w^2-k^2}(b-a)=1 \quad ,\sqrt{w^2-k^2}(b-a)=2n\pi,$$

$$k_n^2=w^2-\left(\frac{2n\pi}{b-a}\right)^2, n=1,2,3,...... \tag{2.17}$$

and hence

$$\varphi_0=\exp(ik_nx)(c_1\sin(\frac{2n\pi}{b-a})y+c_2\cos(\frac{2n\pi}{b-a})\,y) \tag{2.18}$$

Substituting (2.18) into (2.8), we have

$$\varphi_1(x,a)-\varphi_1(x,b)=\delta_1\exp(i(k_n+k_w))\,x+\delta_2\exp(i(k_n-k_w))x \tag{2.19}$$

where

$$\left.\begin{aligned}\delta_1&=\frac{1}{2i}(\frac{2n\pi}{b-a})(-(\cos(\frac{2n\pi}{b-a})a-\cos(\frac{2n\pi}{b-a})b)+(\sin(\frac{2n\pi}{b-a})a-\sin(\frac{2n\pi}{b-a})b)\\ \delta_2&=\frac{1}{2i}(\frac{2n\pi}{b-a})((\cos(\frac{2n\pi}{b-a})a-\cos(\frac{2n\pi}{b-a})b)-(\sin(\frac{2n\pi}{b-a})a-\sin(\frac{2n\pi}{b-a})b)\end{aligned}\right\} \tag{2.20}$$

To find the solution of (2.6,2.7 and 2.19) for φ_1, we note that the boundary conditions (2.19) is nonhomogenous, then the variables be separated as follows

$$\varphi_1=\phi_1(y)\exp(i(k_n+k_w))\,x+\phi_2(y)\exp(i(k_n-k_w))\,x \tag{2.21}$$

Put (2.21) in (2.6, 2.7 and 2.19), we have

$$(\phi_1''(y)+\alpha_1^2\phi_1(y))\exp(i(k_n+k_w))x+(\phi_2''(y)+\alpha_2^2\phi_2(y))\exp(i(k_n-k_w))x=0$$

where

$$\alpha_1^2=w^2-(k_n+k_w)^2 \ ,\alpha_2^2=w^2-(k_n-k_w)^2 \tag{2.22}$$

$$(\phi_1'(a)-\phi_1'(b))\exp(i(k_n+k_w))\,x+(\phi_2'(a)-\phi_2'(b))\exp(i(k_n-k_w))\,x=0$$

$$\phi_1(a)\exp(i(k_n+k_w))\,x+\phi_2(a)\exp(i(k_n-k_w))\,x-\phi_1(b)\exp(i(k_n+k_w))x-$$

$$-\phi_2(b)\exp(i(k_n-k_w))x=\delta_1\exp(i(k_n+k_w))x+\delta_2\exp(i(k_n-k_w))x$$

Equating the coefficients of each of exponentials on both sides, we obtain

$$\phi_1''(y)+\alpha_1^2\,\phi_1(y)=0\ ,\phi_1'(a)-\phi_1'(b)=0\ ,\phi_1(a)-\phi_1(b)=\delta_1 \qquad (2.23)$$

$$\phi_2''(y)+\alpha_2^2\phi_2(y)=0\,,\phi_2'(a)-\phi_2'(b)=0\,,\phi_2(a)-\phi_2(b)=\delta_2 \qquad (2.24)$$

The general solution of (2.23) is

$\phi_1=A_1\cos\alpha_1 y+A_2\sin\alpha_1 y$, and substituting in boundary conditions we have

$$A_1(-\alpha_1\sin\alpha_1 a+\alpha_1\sin\alpha_1 b)+A_2(\alpha_1\cos\alpha_1 a-\alpha_1\cos\alpha_1 b)=0$$

$$A_1(\cos\alpha_1 a-\cos\alpha_1 b)+A_2(\sin\alpha_1 a-\sin\alpha_1 b)=\delta_1$$

hence

$$\phi_1=\frac{\delta_1(\cos\alpha_1 a-\cos\alpha_1 b)}{2(1-\cos\alpha_1(b-a))}\cos\alpha_1 y\ +\frac{\delta_1(\sin\alpha_1 a-\sin\alpha_1 b)}{2(1-\cos\alpha_1(b-a))}\sin\alpha_1 y$$

Similarly the solution of (2.24) is

$$\phi_2=\frac{\delta_2(\cos\alpha_2 a-\cos\alpha_2 b)}{2(1-\cos\alpha_2(b-a))}\cos\alpha_2 y\ +\frac{\delta_2(\sin\alpha_2 a-\sin\alpha_2 b)}{2(1-\cos\alpha_2(b-a))}\sin\alpha_2 y$$

Therefore from (2.21), we have

$$\begin{aligned}\varphi_1=&\Big(\frac{\delta_1(\cos\alpha_1 a-\cos\alpha_1 b)}{2(1-\cos\alpha_1(b-a))}\ \cos\alpha_1\, y\ +\\ &+\frac{\delta_1(\sin\alpha_1 a-\sin\alpha_1 b)}{2(1-\cos\alpha_1(b-a))}\ \sin\alpha_1 y\Big)\exp(i(k_n+k_w))\,x\ +\\ &+\Big(\frac{\delta_2(\cos\alpha_2 a-\cos\alpha_2 b)}{2(1-\cos\alpha_2(b-a))}\ \cos\alpha_2\ y\ +\\ &+\frac{\delta_2(\sin\alpha_2 a-\sin\alpha_2 b)}{2(1-\cos\alpha_2(b-a))}\ \sin\alpha_2\ y\Big)\exp(i(k_n-k_w))\,x\end{aligned} \qquad (2.25)$$

Substituting (2.18, 2.25) into (2.1), we obtain

$$\begin{aligned}\varphi=&\exp(ikx)(\sin\frac{2n\pi}{b-a}y+\cos\frac{2n\pi}{b-a}y)+\varepsilon\Big(\Big(\frac{\delta_1(\cos\alpha_1 a-\cos\alpha_1 b)}{2(1-\cos\alpha_1(b-a))}\cos\alpha_1 y+\\ &+\frac{\delta_1(\sin\alpha_1 a-\sin\alpha_1 b)}{2(1-\cos\alpha_1(b-a))}\sin\alpha_1 y\Big)\exp(i(k_n+k_w))x+\\ &+\Big(\frac{\delta_2(\cos\alpha_2 a-\cos\alpha_2 b)}{2(1-\cos\alpha_2(b-a))}\cos\alpha_2 y\ +\\ &+\frac{\delta_2(\sin\alpha_2 a-\sin\alpha_2 b)}{2(1-\cos\alpha_2(b-a))}\sin\alpha_2 y\Big)\exp(i(k_n-k_w))x\Big)+\ldots\ldots\end{aligned} \qquad (2.26)$$

If either $\cos\alpha_1(b-a)=1$ *or* $\cos\alpha_2(b-a)=1$, the second term tends to infinity, and therefore the series (2.26) is nonuniform.

Since $\cos\alpha_1(b-a)=1 \Rightarrow \alpha_1=\frac{2m\pi}{b-a}, m=0,1,2,......$

$$w^2-(k_n+k_w)^2 \approx (\frac{2m\pi}{b-a})^2 \quad or \quad w^2-(k_n-k_w)^2 \approx (\frac{2m\pi}{b-a})^2 \tag{2.27}$$

But $w^2-(\frac{2m\pi}{b-a})^2=k_m^2$ from (2.17), hence (2.27) can be written as

$$(k_n+k_w)^2 \approx k_m^2 \quad or \quad (k_n-k_w)^2 \approx k_m^2 \quad or \quad k_w=\mp k_n \mp k_m$$

To determine an expansion valid when $k_w \approx k_n-k_m$, let the parameter σ as $k_w=k_n-k_m+\varepsilon\sigma$

III - Using method of multiple scales

Using the method of multiple scales in [5] and seek the expansion in the form

$$\varphi(x,y,\varepsilon)=\varphi(x_0,x_1,y,\varepsilon)=\varphi_0(x_0,x_1,y)+\varepsilon\varphi_1(x_0,x_1,y)+...... \tag{2.28}$$

where $x_0=x$ *and* $x_1=\varepsilon x$, thus

$$\left.\begin{aligned}\frac{\partial}{\partial x}&=\frac{\partial}{\partial x_0}+\varepsilon\frac{\partial}{\partial x_1}+......\\ \frac{\partial^2}{\partial x^2}&=\frac{\partial^2}{\partial x_0^2}+2\varepsilon\frac{\partial^2}{\partial x_0\partial x_1}+......\end{aligned}\right\} \tag{2.29}$$

Substituting (2.28, 2.29) in (1.1,1.2 and 2.2) and equating coefficients of likes power of ε, we obtain

order ε^0

$$\left.\begin{aligned}&\varphi_{0x_0x_0}+\varphi_{0yy}+w^2\varphi_0=0\\ &\varphi_{0y}(x_0,x_1,a)-\varphi_{0y}(x_0,x_1,b)=0\\ &\varphi_0(x_0,x_1,a)-\varphi_0(x_0,x_1,b)=0\end{aligned}\right\} \tag{2.30}$$

order ε^1

$$\varphi_{1x_0x_0}+\varphi_{1yy}+w^2\varphi_1=-2\varphi_{0x_0x_1} \tag{2.31}$$

$$\varphi_{1y}(x_0,x_1,a)-\varphi_{1y}(x_0,x_1,b)=0 \tag{2.32}$$

$$\varphi_1(x_0,x_1,a)-\varphi_1(x_0,x_1,b)=-\varphi_{0y}(x_0,x,a)\sin k_w x_0+\varphi_{0y}(x_0,x,b)\sin k_w x_0 \tag{2.33}$$

The solution of (2.30) can be obtained by separating variables, however instead of making φ_0 contain only one mode, we make φ_0 contain the two modes, the mth and nth mode and hence can be written as

$$\begin{aligned}\varphi_0 = {} & A_n(x_1)\,(\cos\frac{2n\pi}{b-a}y+\sin\frac{2n\pi}{b-a}y)\,\exp(ik_nx_0)\, + \\ & + A_m(x_1)\,(\cos\frac{2m\pi}{b-a}y+\sin\frac{2m\pi}{b-a}y)\,\exp(ik_mx_0)\end{aligned} \tag{2.34}$$

where k_n, k_m are defined by (2.17) and A_n, A_m will be determined.

Substituting (2.34) into (2.31, 2.33), we have

$$\frac{\partial^2\varphi_1}{\partial x_0^2}+\frac{\partial^2\varphi_1}{\partial y^2}+w^2\varphi_1=-2ik_nA_n'(x_1)(\cos\frac{2n\pi}{b-a}y+\sin\frac{2n\pi}{b-a}y)\exp(ik_nx_0)-$$

$$-2ik_mA_m'(x_1)(\cos\frac{2m\pi}{b-a}y+\sin\frac{2m\pi}{b-a}y)\exp(ik_mx_0) \qquad (2.35)$$

$$\varphi_1(x_0,x_1,a)-\varphi_1(x_0,x_1,b)=-A_n(x_1)(\frac{2n\pi}{b-a})(-\sin\frac{2n\pi}{b-a}a \;+$$

$$+\cos\frac{2n\pi}{b-a}a)\sin(k_wx_0)\exp(ik_nx_0)-A_m(x_1)(\frac{2m\pi}{b-a})(-\sin\frac{2m\pi}{b-a}a \;+$$

$$+\cos\frac{2m\pi}{b-a}a)\sin(k_wx_0)\exp(ik_mx_0)+A_n(x_1)(\frac{2n\pi}{b-a})(-\sin\frac{2n\pi}{b-a}b \;+$$

$$+\cos\frac{2n\pi}{b-a}b)\sin(k_wx_0)\exp(ik_nx_0)+A_m(x_1)(\frac{2m\pi}{b-a})(-\sin\frac{2m\pi}{b-a}b \;+$$

$$+\cos\frac{2m\pi}{b-a}b)\sin(k_wx_0)\exp(ik_mx_0)$$

$$\varphi_1(x_0,x_1,a)-\varphi_1(x_0,x_1,b)=\delta_{1n}\,A_n\exp(i(k_n+k_w))\,x_0\;+$$

$$+\delta_{2n}\,A_n\,\exp(i(k_n-k_w))\,x_0\;+$$

$$+\delta_{1m}A_m\exp(i(k_m+k_w))x_0+\delta_{2m}A_m\exp(i(k_m-k_w))x_0 \qquad (2.36)$$

where δ_{1n},δ_{2n} are defined by (2.20) and δ_{1m},δ_{2m} are also defined by (2.20) if n is replaced by m.

To determine the solvability conditions for the first problem, substitute $k_w=k_n-k_m+\varepsilon\sigma$ into (2.36), we have

$$\varphi_1(x_0,x_1,a)-\varphi_1(x_0,x_1,b)=\delta_{1n}A_n\exp(i(2k_n-k_m+\varepsilon\sigma))x_0\;+$$

$$+\;\delta_{2n}A_n\exp(i(k_m-\varepsilon\sigma))x_0+\delta_{1m}A_m\exp(i(k_n+\varepsilon\sigma))x_0\;+$$

$$+\;\delta_{2m}A_m\exp(i(2k_m-k_n-\varepsilon\sigma))x_0$$

$$=\delta_{1n}A_n\exp(i\sigma x_1)\exp(i(2k_n-k_m))x_0+\delta_{2n}A_n\exp(-i\sigma x_1)\exp(ik_mx_0)+$$

$$+\delta_{1m}A_m\exp(ik_nx_0)\exp(i\sigma x_1)+\delta_{2m}A_m\exp(-i\sigma x_1)\exp(i(2k_m-k_n))x_0$$

(2.37)

where $x_1=\varepsilon x_0$

We note that the terms $\exp(ik_mx_0)\,and\,\exp(ik_nx_0)$ in (2.35) and (2.37) may lead to incompatibilities and solvability conditions must be imposed on them. These solvability conditions can be obtained by seeking a particular solution corresponding to these terms in the form

$$\varphi_1 = \phi_n(x_1, y)\exp(ik_n x_0) + \phi_m(x_1, y)\exp(ik_m x_0) \tag{2.38}$$

Substituting (2.38) into (2.32, 2.35 and 2.37) and equating the coefficients of $\exp(ik_n x_0)$ *and* $\exp(ik_m x_0)$ on both sides, we have

$$\left.\begin{aligned} &\frac{\partial^2\phi_n}{\partial y^2} + \left(\frac{2n\pi}{b-a}\right)^2\phi_n = -2ik_n A_n'\left(\cos\frac{2n\pi}{b-a}y + \sin\frac{2n\pi}{b-a}y\right) \\ &\phi_{ny}(x_1, a) - \phi_{ny}(x_1, b) = 0 \\ &\phi_n(x_1, a) - \phi_n(x_1, b) = \delta_{1m} A_m \exp(i\sigma x_1) \end{aligned}\right\} \tag{2.39}$$

$$\left.\begin{aligned} &\frac{\partial^2\phi_m}{\partial y^2} + \left(\frac{2m\pi}{b-a}\right)^2\phi_m = -2ik_m A_m'\left(\cos\frac{2m\pi}{b-a}y + \sin\frac{2m\pi}{b-a}y\right) \\ &\phi_{my}(x_1, a) - \phi_{my}(x_1, b) = 0 \\ &\phi_m(x_1, a) - \phi_m(x_1, b) = \delta_{2n} A_n \exp(-i\sigma x_1) \end{aligned}\right\} \tag{2.40}$$

Thus, determining the solvability conditions ϕ_1 has been transformed into determining the solvability conditions for ϕ_n *and* ϕ_m.

We note that the equation in (2.39) is self - adjoint, the solution of the adjoint problem can be taken as

$$u = \cos\frac{2n\pi}{b-a}y + \sin\frac{2n\pi}{b-a}y$$

Multiplying the equation in (2.39) by $u(y)$ and integrating the result by parts from $y = a$ *to* $y = b$ to transfer the derivatives from ϕ_n *to* u, we obtain

$$\int_a^b u\left(\frac{\partial^2\phi_n}{\partial y^2} + \left(\frac{2n\pi}{b-a}\right)^2\phi_n\right)dy = \int_a^b \left(-2ik_n A_n'\left(\cos\frac{2n\pi}{b-a}y + \sin\frac{2n\pi}{b-a}y\right)^2\right)dy$$

$$\int_a^b \left(u'' + \left(\frac{2n\pi}{b-a}\right)^2 u\right)\phi_n dy + \left|\frac{\partial\phi_n}{\partial y}u - \phi_n u'\right|_a^b = -2ik_n A_n'(b-a) \tag{2.41}$$

To find the boundary conditions for adjoint problem, put the right side in (2.41) equal to zero and using the homogenous boundary conditions in (2.39), we obtain

$$\frac{\partial\phi_n(b)}{\partial y}(u(b) - u(a)) - \phi_n(b)(u'(b) - u'(a)) = 0$$

Equating the coefficients of $\frac{\partial\phi_n(b)}{\partial y}$ *and* $\phi_n(b)$ by zero, we have

$$u(b) = u(a) \quad , u'(b) = u'(a) \tag{2.42}$$

which are the boundary conditions for adjoint problem.

Now from (2.41) and using the nonhomogenous conditions in (2.39) and boundary conditions (2.42), we obtain

$$-\phi_n(x)\,u'(x)\,\Big|_a^b = -2ik_n A_n'\,(b-a)$$

$$-(\phi_n(b)u'(b)-\phi_n(a)u'(a)) = -2ik_n A_n'\,(b-a)$$

or

$$\delta_{1m} A_m \exp(i\sigma\, x_1)(-\sin\frac{2n\pi}{b-a}b+\cos\frac{2n\pi}{b-a}b)\frac{2n\pi}{b-a} = -2ik_n A_n'(b-a)$$

therefore

$$A_n' = (\delta_{1m} A_m \exp(i\sigma\, x_1)(\cos\frac{2n\pi}{b-a}b-\sin\frac{2n\pi}{b-a}b)\frac{2n\pi}{b-a})\frac{ik_n^{-1}}{2(b-a)} \qquad (2.43)$$

and this is the solvability condition for problem (2.39).

Similarly if $m \neq 0$, the solvability condition for problem (2.40)

$$A_m' = (\delta_{2n} A_n \exp(i\sigma\, x_1)(\cos\frac{2m\pi}{b-a}b-\sin\frac{2m\pi}{b-a}b)\frac{2m\pi}{b-a})\frac{ik_m^{-1}}{2(b-a)} \qquad (2.44)$$

If we let $A_n = a_n \exp(i\gamma_1 x_1)\ , A_m = a_m \exp(i\gamma_2 x_1)$ (2.45)

where $a_n\ , a_m , \gamma_1\ and\ \gamma_2$ are constants, then it follows from (2.43) and (2.44)

$$i\gamma_1 a_n = (\delta_{1m} a_m (\cos\frac{2n\pi}{b-a}b-\sin\frac{2n\pi}{b-a}b)\frac{2n\pi}{b-a})\frac{ik_n^{-1}}{2(b-a)} \qquad (2.46)$$

$$i\gamma_2 a_m = (\delta_{2n} a_n (\cos\frac{2m\pi}{b-a}b-\sin\frac{2m\pi}{b-a}b)\frac{2m\pi}{b-a})\frac{ik_m^{-1}}{2(b-a)} \qquad (2.47)$$

$$\gamma_2 = \gamma_1 - \sigma \qquad (2.48)$$

Eliminating γ_2 *and* a_m from (2.46) by using (2.48), we have

$$\gamma_1(\gamma_1-\sigma) = (\delta_{1m}\delta_{2n}(\cos\frac{2n\pi}{b-a}b-\sin\frac{2n\pi}{b-a}b)(\cos\frac{2m\pi}{b-a}b-\sin\frac{2m\pi}{b-a}b)$$
$$(\frac{2n\pi}{b-a})(\frac{2m\pi}{b-a})(\frac{k_n^{-1}}{2(b-a)})(\frac{k_m^{-1}}{2(b-a)})$$

or

$$\gamma_1^{\ 2} - \sigma\gamma_1 - (\delta_{1m}\delta_{2n}(\cos\frac{2n\pi}{b-a}b-\sin\frac{2n\pi}{b-a}b)(\cos\frac{2m\pi}{b-a}b-\sin\frac{2m\pi}{b-a}b)$$
$$(\frac{2n\pi}{b-a})(\frac{2m\pi}{b-a})(\frac{k_n^{-1}k_m^{-1}}{4(b-a)^2}) = 0$$

$$\gamma_1 = \frac{1}{2}\sigma \mp \frac{1}{2}(\sigma^2 + \delta_{1m}\delta_{2n}(\cos\frac{2n\pi}{b-a}b - \sin\frac{2n\pi}{b-a}b)(\cos\frac{2m\pi}{b-a}b - \sin\frac{2m\pi}{b-a}b)$$
$$(\frac{2n\pi}{b-a})(\frac{2m\pi}{b-a})(\frac{k_n^{-1}k_m^{-1}}{b-a}))^{\frac{1}{2}}$$

When substitute the value of γ_1 into (2.45), we find the values of A_n , A_m and then substitute in (2.43), (2.44), we obtain the solvability conditions of the problems (2.39), (2.40).

3. Conclusions

The perturbation method has been applied to find solvability conditions of the 2nd order boundary value problem (1.1-1.3) which are

$$A_n' = (\delta_{1m}A_m \exp(i\sigma x_1)(\cos\frac{2n\pi}{b-a}b - \sin\frac{2n\pi}{b-a}b)\frac{2n\pi}{b-a})\frac{ik_n^{-1}}{2(b-a)}$$
$$A_m' = (\delta_{2n}A_n \exp(i\sigma x_1)(\cos\frac{2m\pi}{b-a}b - \sin\frac{2m\pi}{b-a}b)\frac{2m\pi}{b-a})\frac{ik_m^{-1}}{2(b-a)}$$

These conditions were satisfied.

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