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Studying Stability of a Non-linear Autoregressive Model

Anas Salim Youns **Abdul Ghafoor Jasim Salim**

College of Basic Education *College of Computers Sciences*
Department of Mathematics *Department of Mathematics*
University of Mosul/Mosul/Iraq *University of Mosul/Mosul/Iraq*

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الملخص

تم في هذا البحث دراسة الاستقرارية لأحد نماذج الانحدار الذاتي غير الخطي بدوال مثلثية زائدية متعدد حدود بدالة الظل الزائدية (النقطة المنفردة وشروط استقراريتها واستقرارية دورة النهاية) وباستخدام طريقة التقريب الخطية المحلية

(local linearization approximation method) ومن ثم طبقنا تلك النتائج النظرية باستخدام بعض الأمثلة لتوضيح ذلك.

Abstract

In this paper we study the stability of one non-linear autoregressive model with hyperbolic triangle function polynomial with hyperbolic tangent function (singular point and it's stability conditions and the stability of the limit cycle) by using the local linearization approximation method and we apply this theory results by using some of examples to explain that.

Key words. *Non-linear time series model; Non-linear random vibration; Autoregressive model; Limit cycle; Singular point; Stability*

Introduction

In the field of discrete time non-linear time series modeling, there are many different types of a non-linear model. In [1] Ozaki proposed the method of local linearization approximation to find the stability of a non-linear exponential autoregressive models.

In this paper we study the stability of a non-linear autoregressive model with hyperbolic tangent function by using the local linear approximation method which is used in order to study such models to find the stability condition (singular point and the stability condition for it and the limit cycle) and we give some examples to explain this method.

2.Basic concepts of time series

Definition 2.1: A difference equation of order n over the set of k-values 0, 1, 2, ... is an equation of the form $F(k, y_k, y_{k-1}, \ldots\ldots, y_{k-n}) = 0$.

Where F is a given function, n is some positive integer, and k = 0, 1, 2, … [2].

Definition 2.2: A time series is a set of observations measured sequentially through time. These measurements may be made continuously through time or be taken at a discrete set of time points. Then a time series is a sequence of random variables defined on probability space refered by index (t) that belongs to index set T. We refer to time series by $\{X_t; -\infty < t < \infty\}$ if T take a continuous values, or $\{X_t; t = 0, \pm1, \pm2, \ldots\ldots\}$ if T takes a discrete values [3].

Definition 2.3: A time series $\{X_t\}$ is represented by a linear autoregressive model if it satisfies the following difference equation:

$$X_t + a_1 X_{t-1} + a_2 X_{t-2} + \ldots + a_p X_{t-p} = Z_t$$

Where $\{Z_t\}$ is a white noise and $a_1, a_2, \ldots\ldots\ldots, a_p$ are real constants [4].

Definition 2.4:The exponential autoregressive model of order p, EXPAR(P) is defined by the following equation

$$X_t = \sum_{j=1}^{p} \left(\phi_j + \pi_j e^{-X_{t-1}^2}\right) X_{t-j} + Z_t$$

Where $\{Z_t\}$ is a white noise and $\phi_1,......., \phi_p; \pi_1,........, \pi_p$ are the parameters of the model [5].

Definition 2.5: The bilinear model of order (p,q,m,s) satisfies the equation

$$X_t = c + \sum_{i=1}^{p} \phi_i X_{t-i} - \sum_{j=1}^{q} \theta_j Z_{t-j} + \sum_{i=1}^{m} \sum_{j=1}^{s} \beta_{ij} X_{t-i} Z_{t-j} + Z_t$$

Where p , q , m and s are nonnegative (or integer constants), and $\{Z_t\}$ is a sequence of independent identically distributed random variables and $\phi_1,......., \phi_p; \theta_1,........, \theta_q; \beta_{ij}; \forall i = 1,.....m, \forall j = 1,......s$ are the parameters of the model [6].

Definition 2.6: A singular point of $X_t = f(X_{t-1}, X_{t-2},......, X_{t-p})$ is defined as a point ζ which every trajectory of $X_t = f(X_{t-1}, X_{t-2},......, X_{t-p})$ beginning sufficiently approaches near it either for $t \to \infty$ or for $t \to -\infty$. If it approaches it for $t \to \infty$ we call it stable singular point and if it approaches it for $t \to -\infty$ we call it unstable singular point.

Obviously a singular point ζ satisfies $\zeta = f(\zeta)$ [7].

Definition 2.7: A limit cycle of $X_t = f(X_{t-1}, X_{t-2},......, X_{t-p})$ is defined as an isolated and closed trajectory $X_{t+1}, X_{t+2},......, X_{t+q}$ where q is a positive integer. Closed means that if initial values $(X_1,....., X_p)$ belong to the limit cycle then $(X_{1+k_q}, X_{2+k_q},......, X_{p+k_q}) = (X_1,......., X_p)$ for any integer k. Isolated means that every trajectory beginning sufficiently near the limit cycle approaches it either for $t \to \infty$ or for $t \to -\infty$. If it approaches it for $t \to \infty$ we call it stable limit cycle and if it approaches it for $t \to -\infty$ we call it unstable limit cycle [7].

Theorem 1:

Let $\{X_t\}$ be expressed by the exponential autoregressive model

$$X_t = \left(\phi_1 + \pi_1 e^{-X_{t-1}^2}\right) X_{t-1} + Z_t$$

A limit cycle of period q , $X_{t+1}, X_{t+2}, \ldots\ldots, X_{t+q}$ of the model is orbitally stable if $\left|\frac{\zeta_{t+q}}{\zeta_t}\right| < 1$.

For proof see [7].

The proposed model: A non-linear autoregressive model (polynomial with hyperbolic tangent function)) of order p is defined by

$$X_t = \sum_{i=1}^{p} [\phi_i \tanh(X_{t-1})]^i X_{t-i} + Z_t$$

Where $\{Z_t\}$ is a white noise process and $\phi_1, \ldots\ldots, \phi_p$ are the parameters(real constant) of the model.

3.The stability of the proposed model

In this section, by using the local linear approximation method, we shall study the stability of a non-linear autoregressive model with a hyperbolic tangent function with low order such that p=1,2,3 .

3.1.1 The singular point

Let we have the following model

$$X_t = \sum_{i=1}^{p} [\phi_i \tanh(X_{t-1})]^i X_{t-i} + Z_t \qquad (1)$$

Let p=1 , then we have

$$X_t = [\phi_1 \tanh X_{t-1}] X_{t-1} + Z_t \qquad (2)$$

Suppose that the white noise has not effect (Z_t be minimum, i.e. $Z_t = 0$) to get a deterministic model which has a limit cycle, and using $\zeta = f(\zeta)$, we get the singular point ζ as:

$$\zeta = [\phi_1 \tanh(\zeta)]\zeta$$

Or

$$\zeta = \tanh^{-1}(\frac{1}{\phi_1}), \left|\tfrac{1}{\phi_1}\right| < 1, (\zeta \neq 0), (\phi_1 \neq 0;1;-1)$$

Or equivalently $\zeta = \frac{1}{2}\ln\{(\frac{\phi_1+1}{\phi_1-1})\}$ (3)

Therefore, the non-zero singular point is exists if $(\frac{\phi_1+1}{\phi_1-1}) > 0$, $(\phi_1 \neq 0;1;-1)$.

3.1.2 The stability of singular point:

We will find the stability condition for the non-zero singular point as follows :

Put $X_s = \zeta + \zeta_s$ for s=t,t-1, in equation (2)(when p=1), and also suppose that the white noise has not effect, then we have:

$$\zeta + \zeta_t = \phi_1[\tanh(\zeta + \zeta_{t-1})](\zeta + \zeta_{t-1}) \quad (4)$$

Then $\zeta_t = [\frac{2\phi_1\zeta e^{2\zeta} + \phi_1(e^{2\zeta}-1) - 2\zeta e^{2\zeta})}{(e^{2\zeta}+1)}]\zeta_{t-1}$ (5)

Or $\zeta_t = h_1\zeta_{t-1}$, where $h_1 = \frac{2\phi_1\zeta e^{2\zeta} + \phi_1(e^{2\zeta}-1) - 2\zeta e^{2\zeta})}{(e^{2\zeta}+1)}$ (6)

Equation (6) is a first order linear autoregressive model which is stable if the root λ_1 of the characteristic equation lies inside the unit circle.

i.e. if $|\lambda_1| = |h_1| < 1$.

3.1.3 The limit cycle:

We find the stability condition of a limit cycle (if it exists) as follows:

Let the limit cycle of period q of the proposed model in the equation (2) have the form $X_t, X_{t+1}, X_{t+2}, \ldots\ldots, X_{t+q} = X_t$ .The points X_s near the limit cycle are represented as $X_s = X_s + \zeta_s$, and the same note on $\{Z_t\}$ when we find the singular point, then we have

$$X_t + \zeta_t = [\phi_1 \frac{\left(e^{(x_{t-1}+\zeta_{t-1})} - e^{-(x_{t-1}+\zeta_{t-1})}\right)}{\left(e^{(x_{t-1}+\zeta_{t-1})} + e^{-(x_{t-1}+\zeta_{t-1})}\right)}](X_{t-1} + \zeta_{t-1}) \quad (7)$$

Therefore

$$\zeta_t = [\frac{2\phi_1 x_{t-1} e^{2x_{t-1}} + \phi_1(e^{2x_{t-1}}-1) - 2e^{2x_{t-1}}x_t)}{(e^{2x_{t-1}}+1)}]\zeta_{t-1} \quad (8)$$

Equation (8) is a linear difference equation with a periodic coefficient, which is difficult to solve analytically, what we want to know whether ζ_t of (8) converges to zero or not, and this can be checked by seeing whether $\left|\frac{\zeta_{t+q}}{\xi_t}\right|$ is less than one or not [1].

Let t=t+q in equation (8).

Then $$\zeta_{t+q} = [\frac{2\phi_1 x_{t+q-1} e^{2x_{t+q-1}} + \phi_1(e^{2x_{t+q-1}} - 1) - 2e^{2x_{t+q-1}} x_{t+q})}{(e^{2x_{t+q-1}} + 1)}]\zeta_{t+q-1} \quad (9)$$

Or $$\zeta_{t+q} = \prod_{i=1}^{q}[\frac{2\phi_1 x_{t+i-1} e^{2x_{t+i-1}} + \phi_1(e^{2x_{t+i-1}} - 1) - 2e^{2x_{t+i-1}} x_{t+i})}{(e^{2x_{t+i-1}} + 1)}]\zeta_t \quad (10)$$

Then equation (10) is orbitally stable if $\left|\frac{\zeta_{t+q}}{\zeta_t}\right| < 1$ (theorem(1)).

Therefore, the limit cycle of the propose model is orbitally stable if

$$\left|\frac{\zeta_{t+q}}{\zeta_t}\right| = \left|\prod_{i=1}^{q}[\frac{2\phi_1 x_{t+i-1} e^{2x_{t+i-1}} + \phi_1(e^{2x_{t+i-1}} - 1) - 2e^{2x_{t+i-1}} x_{t+i})}{(e^{2x_{t+i-1}} + 1)}]\right| < 1 \quad (11)$$

3.2.1 The singular point

For p=2 , we have

$$X_t = [\phi_1 \tanh X_{t-1}]X_{t-1} + [\phi_2^2 \tanh^2 X_{t-1}]X_{t-2} + Z_t \quad (12)$$

Suppose that $Z_t = 0$, and $\zeta = f(\zeta)$, we get

$$\zeta = [\phi_1 \tanh(\zeta)]\zeta + [\phi_2^2 \tanh^2(\zeta)]\zeta$$

Since $\zeta \neq 0$, then we divide by it to get:

$\phi_1 \tanh(\zeta) + \phi_2^2 \tanh^2(\zeta) - 1 = 0$, and also $\tanh(\zeta) = (\frac{-\phi_1 \mp \sqrt{\phi_1^2 + 4\phi_2^2}}{2\phi_2^2})$

The singular points of the model in equation(12) are

$$\zeta = \tanh^{-1}(\frac{-\phi_1 \mp \sqrt{\phi_1^2 + 4\phi_2^2}}{2\phi_2^2}) \quad (13)$$

3.2.2 The stability of singular point:

We will find the stability condition for the non-zero singular points of equation(12) as follows:

$$\zeta_t = [\frac{4\phi_1\zeta e^{4\zeta} + \phi_1(e^{4\zeta}-1) - 4\zeta e^{2\zeta}(e^{2\zeta}+1) + \phi_2^2(4\zeta e^{2\zeta}(e^{2\zeta}-1))}{(e^{2\zeta}+1)^2}]\zeta_{t-1} + [\frac{\phi_2^2(e^{2\zeta}-1)^2}{(e^{2\zeta}+1)^2}]\zeta_{t-2}$$

Or

$$\zeta_t = h_1\zeta_{t-1} + h_2\zeta_{t-2} \qquad (14)$$

Where

$$h_1 = [\frac{4\phi_1\zeta e^{4\zeta} + \phi_1(e^{4\zeta}-1) - 4\zeta e^{2\zeta}(e^{2\zeta}+1) + \phi_2^2(4\zeta e^{2\zeta}(e^{2\zeta}-1))}{(e^{2\zeta}+1)^2}]$$

$$h_2 = [\frac{\phi_2^2(e^{2\zeta}-1)^2}{(e^{2\zeta}+1)^2}]$$

Then from the comparison between the roots of the equation (14) and its a linear model of order two which have characteristic equation of the form,

$$v^2 - h_1 v - h_2 = 0 = (v-\lambda_1)(v-\lambda_2) = v^2 - (\lambda_1+\lambda_2)v + \lambda_1\lambda_2$$

Then $h_1 = (\lambda_1+\lambda_2), h_2 = -\lambda_1\lambda_2$

Where λ_1, λ_2 are the roots of the characteristic equation of the model.

The stability condition is that $|\lambda_i| < 1$; for all i =1,2 .

3.2.3 The limit cycle:

Let the limit cycle (for the 3rd-order) in equation (12) has the form $X_t, X_{t+1}, X_{t+2}, \ldots\ldots, X_{t+q} = X_t$ .The points X_s near the limit cycle is represented as $X_s = X_s + \zeta_s, \forall s = t, t-1, t-2, t-3$ and the same note on $\{Z_t\}$, then we have

$$X_t + \zeta_t = [\phi_1(\frac{e^{x_{t-1}+\zeta_{t-1}} - e^{-(x_{t-1}+\zeta_{t-1})}}{e^{(x_{t-1}+\zeta_{t-1})} + e^{-(x_{t-1}+\zeta_{t-1})}})](X_{t-1} + \zeta_{t-1}) + [\phi_2^2(\frac{e^{x_{t-1}+\zeta_{t-1}} - e^{-(x_{t-1}+\zeta_{t-1})}}{e^{x_{t-1}+\zeta_{t-1}} + e^{-(x_{t-1}+\zeta_{t-1})}})^2](X_{t-2} + \zeta_{t-2}) \qquad (15)$$

Then, by using maclaurin series expansion for the exponential function we get

$$\zeta_t = [\frac{\phi_1(4x_{t-1}e^{4x_{t-1}} + e^{4x_{t-1}} - 1) + \phi_2^2(4x_{t-2}e^{2x_{t-1}}(e^{2x_{t-1}} - 1)) - 4x_t e^{2x_{t-1}}(e^{2x_{t-1}} + 1)}{(e^{2x_{t-1}} + 1)^2}]\zeta_{t-1} + [\frac{\phi_2^2(e^{2x_{t-1}} - 1)^2}{(e^{2x_{t-1}} + 1)^2}]\zeta_{t-2} \quad (16)$$

Then, we checked whether $\left|\frac{\zeta_{t+q}}{\xi_t}\right| < 1$ or not.

$$\zeta_{t+q} = [\frac{\phi_1(4x_{t+q-1}e^{4x_{t+q-1}} + e^{4x_{t+q-1}} - 1) + \phi_2^2(4x_{t+q-2}e^{2x_{t+q-1}}(e^{2x_{t+q-1}} - 1)) - 4x_{t+q}e^{2x_{t+q-1}}(e^{2x_{t+q-1}} + 1)}{(e^{2x_{t+q-1}} + 1)^2}]\zeta_{t+q-1} + [\frac{\phi_2^2(e^{2x_{t+q-1}} - 1)^2}{(e^{2x_{t+q-1}} + 1)^2}]\zeta_{t+q-2} \quad (17)$$

$$\zeta_{t+q} = \{\prod_{i=1}^{q}[\frac{\phi_1(4x_{t+i-1}e^{4x_{t+i-1}} + e^{4x_{t+i-1}} - 1) + \phi_2^2(4x_{t+i-2}e^{2x_{t+i-1}}(e^{2x_{t+i-1}} - 1)) - 4x_{t+i}e^{2x_{t+i-1}}(e^{2x_{t+i-1}} + 1)}{(e^{2x_{t+i-1}} + 1)^2}] + \prod_{i=2}^{q}[\frac{\phi_2^2(e^{2x_{t+i-1}} - 1)^2}{(e^{2x_{t+i-1}} + 1)^2}]\}\zeta_t \quad (18)$$

Therefore, the limit cycle of the purposed model is orbitally stable if

$$\left|\frac{\zeta_{t+q}}{\zeta_t}\right| = \left|\{\prod_{i=1}^{q}[\frac{\phi_1(4x_{t+i-1}e^{4x_{t+i-1}} + e^{4x_{t+i-1}} - 1) + \phi_2^2(4x_{t+i-2}e^{2x_{t+i-1}}(e^{2x_{t+i-1}} - 1)) - 4x_{t+i}e^{2x_{t+i-1}}(e^{2x_{t+i-1}} + 1)}{(e^{2x_{t+i-1}} + 1)^2}] + \prod_{i=2}^{q}[\frac{\phi_2^2(e^{2x_{t+i-1}} - 1)^2}{(e^{2x_{t+i-1}} + 1)^2}]\}\right| < 1 \quad (19)$$

3.3.1 The singular point

For p=3 , we have

$$X_t = [\phi_1 \tanh X_{t-1}]X_{t-1} + [\phi_2^2 \tanh^2 X_{t-1}]X_{t-2} + [\phi_3^3 \tanh^3 X_{t-1}]X_{t-3} + Z_t \quad (20)$$

Also, suppose that $Z_t = 0$, and $\zeta = f(\zeta)$, we get

$$\zeta = [\phi_1 \tanh(\zeta)]\zeta + [\phi_2^2 \tanh^2(\zeta)]\zeta + [\phi_3^3 \tanh^3(\zeta)]\zeta, \zeta \neq 0 .$$

Therefore, we get a third order algebraic equation and by using reference [8], we have a, b and c are real constants such that $a = \frac{\phi_2^2}{\phi_3^3}, b = \frac{\phi_1}{\phi_3^3}, c = -\frac{1}{\phi_3^3}$

$$q = c - \frac{1}{3}ab + \frac{2}{27}a^3$$

$$\Delta = c^2 + \tfrac{4}{27}b^3 - \tfrac{2}{3}abc - \tfrac{1}{27}a^2b^2 + \tfrac{4}{27}a^3c$$

Case one : $\Delta = 0$

Then we get three real roots and find it by

$$x_1 = -2\sqrt[3]{\tfrac{q}{2}} - \tfrac{a}{3}, x_2 = x_3 = \sqrt[3]{\tfrac{q}{2}} - \tfrac{a}{3}$$

Case two : $\Delta < 0$

Then we get three different real roots and find it by

$$x_{k+1} = \sqrt[6]{16(q^2 - \Delta)}\cos\frac{\cos^{-1}\frac{-q}{\sqrt{q^2-\Delta}} + 2\pi k}{3} - \frac{a}{3}, k = 0,1,2$$

Case three : $\Delta > 0$

Then we get one real root and two complex conjugate roots and find it by

$$x_1 = \sqrt[3]{\tfrac{-q-\sqrt{\Delta}}{2}} + \sqrt[3]{\tfrac{-q+\sqrt{\Delta}}{2}} - \tfrac{a}{3}$$

$$x_2 = -\tfrac{1}{2}\left(\sqrt[3]{\tfrac{-q-\sqrt{\Delta}}{2}} + \sqrt[3]{\tfrac{-q+\sqrt{\Delta}}{2}}\right) - \tfrac{a}{3} + i\tfrac{\sqrt{3}}{2}\left(\sqrt[3]{\tfrac{-q-\sqrt{\Delta}}{2}} - \sqrt[3]{\tfrac{-q+\sqrt{\Delta}}{2}}\right)$$

$$x_3 = -\tfrac{1}{2}\left(\sqrt[3]{\tfrac{-q-\sqrt{\Delta}}{2}} + \sqrt[3]{\tfrac{-q+\sqrt{\Delta}}{2}}\right) - \tfrac{a}{3} - i\tfrac{\sqrt{3}}{2}\left(\sqrt[3]{\tfrac{-q-\sqrt{\Delta}}{2}} - \sqrt[3]{\tfrac{-q+\sqrt{\Delta}}{2}}\right)$$

The singular points of the model in equation (20) are that

$$\zeta = \tanh^{-1}(x_i), \forall i = 1,2,3 \qquad (21)$$

3.3.2 The stability of singular point:

To find the stability condition for the non-zero singular points of equation (20) , (when p=3) as follows:

$$\zeta_t = \left[\frac{-6\zeta e^{2\zeta}(e^{2\zeta}+1)^2 + \phi_1(e^{2\zeta}+1)^2(e^{2\zeta}-1) + 2\phi_1\zeta e^{2\zeta}(3e^{2\zeta}-1)(e^{2\zeta}+1) + \phi_2^2(2\zeta e^{2\zeta}(3e^{2\zeta}+1)(e^{2\zeta}-1)) + \phi_3^3(6\zeta e^{2\zeta}(e^{2\zeta}-1)^2)}{(e^{2\zeta}+1)^3}\right]\zeta_{t-1}$$

$$+\left[\frac{\phi_2^2(e^{2\zeta}-1)^3}{(e^{2\zeta}+1)^3}\right]\zeta_{t-2} + \left[\frac{\phi_3^3(e^{2\zeta}-1)^3}{(e^{2\zeta}+1)^3}\right]\zeta_{t-3} \qquad (22)$$

Or

$\zeta_t = h_1\zeta_{t-1} + h_2\zeta_{t-2} + h_3\zeta_{t-3}$ is a linear model of order three.

Where

$$h_1 = [\frac{-6\zeta e^{2\zeta}(e^{2\zeta}+1)^2+\phi_1(e^{2\zeta}+1)^2(e^{2\zeta}-1)+2\phi_1\zeta e^{2\zeta}(3e^{2\zeta}-1)(e^{2\zeta}+1)+\phi_2^2(2\zeta e^{2\zeta}(3e^{2\zeta}+1)(e^{2\zeta}-1))+\phi_3^3(6\zeta e^{2\zeta}(e^{2\zeta}-1)^2)}{(e^{2\zeta}+1)^3}]$$

$$h_2 = [\frac{\phi_2^2(e^{2\zeta}-1)^3}{(e^{2\zeta}+1)^3}];\, h_3 = [\frac{\phi_3^3(e^{2\zeta}-1)^3}{(e^{2\zeta}+1)^3}]$$

The characteristic equation of linear model is $v^3 - h_1 v^2 - h_2 v - h_3 = 0$

Then

$$h_1 = \lambda_1 + \lambda_2 + \lambda_3, h_2 = -(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3), h_3 = \lambda_1\lambda_2\lambda_3$$

Where $\lambda_1, \lambda_2, \lambda_3$ are the roots of the characteristic equation of the model.

The stability condition is that $\forall i = 1,2,3, |\lambda_i| < 1$.

3.3.3 The limit cycle:

We find the stability condition for the limit cycle (if it exists) as follows:

Let the limit cycle (for the 3rd-order) in equation (20) has the form $X_t, X_{t+1}, X_{t+2}, \ldots\ldots, X_{t+q} = X_t$ .The points X_s near the limit cycle are represented as $X_s = X_s + \zeta_s, \forall s = t, t-1, t-2, t-3$ and the same note on $\{Z_t\}$, then we have

$$\begin{aligned} X_t + \zeta_t = {} & [\phi_1 \frac{\left(e^{(x_{t-1}+\zeta_{t-1})} - e^{-(x_{t-1}+\zeta_{t-1})}\right)}{\left(e^{(x_{t-1}+\zeta_{t-1})} + e^{-(x_{t-1}+\zeta_{t-1})}\right)}]\left(X_{t-1} + \zeta_{t-1}\right) \\ & + [\phi_2^2 (\frac{\left(e^{(x_{t-1}+\zeta_{t-1})} - e^{-(x_{t-1}+\zeta_{t-1})}\right)}{\left(e^{(x_{t-1}+\zeta_{t-1})} + e^{-(x_{t-1}+\zeta_{t-1})}\right)})^2](X_{t-2} + \zeta_{t-2}) \\ & + [\phi_3^3 (\frac{\left(e^{(x_{t-1}+\zeta_{t-1})} - e^{-(x_{t-1}+\zeta_{t-1})}\right)}{\left(e^{(x_{t-1}+\zeta_{t-1})} + e^{-(x_{t-1}+\zeta_{t-1})}\right)})^3](X_{t-3} + \zeta_{t-3}) \end{aligned} \qquad (23)$$

Then

$$\zeta_t = [\frac{-6x_t e^{2x_{t-1}}(e^{2x_{t-1}}+1)^2+\phi_1(e^{2x_{t-1}}+1)^2(e^{2x_{t-1}}-1)+2\phi_1 x_{t-1}e^{2x_{t-1}}(3e^{2x_{t-1}}-1)(e^{2x_{t-1}}+1)+\phi_2^2(2x_{t-2}e^{2x_{t-1}}(3e^{2x_{t-1}}+1)(e^{2x_{t-1}}-1))+\phi_3^3(6x_{t-3}e^{2x_{t-1}}(e^{2x_{t-1}}-1)^2)}{(e^{2x_{t-1}}+1)^3}]\zeta_{t-1}$$

$$+[\frac{\phi_2^2(e^{2x_{t-1}}-1)^3}{(e^{2x_{t-1}}+1)^3}]\zeta_{t-2} + [\frac{\phi_3^3(e^{2x_{t-1}}-1)^3}{(e^{2x_{t-1}}+1)^3}]\zeta_{t-3} \qquad (24)$$

$$\zeta_{t+q}=[\frac{-6x_{t+q}e^{2x_{t+q-1}}(e^{2x_{t+q-1}}+1)^2+\phi_1(e^{2x_{t+q-1}}+1)^2(e^{2x_{t+q-1}}-1)+2\phi_1x_{t+q-1}e^{2x_{t+q-1}}(3e^{2x_{t+q-1}}-1)(e^{2x_{t+q-1}}+1)+\phi_2^2(2x_{t+q-2}e^{2x_{t+q-1}}(3e^{2x_{t+q-1}}+1)(e^{2x_{t+q-1}}-1))+\phi_3^3(6x_{t+q-3}e^{2x_{t+q-1}}(e^{2x_{t+q-1}}-1)^2)}{(e^{2x_{t+q-1}}+1)^3}]\zeta_{t+q-1}$$

$$+[\frac{\phi_2^2(e^{2x_{t+q-1}}-1)^3}{(e^{2x_{t+q-1}}+1)^3}]\zeta_{t+q-2}+[\frac{\phi_3^3(e^{2x_{t+q-1}}-1)^3}{(e^{2x_{t+q-1}}+1)^3}]\zeta_{t+q-3} \qquad (25)$$

$$\zeta_{t+q}=\prod_{i=1}^{q}[\frac{-6x_{t+i}e^{2x_{t+i-1}}(e^{2x_{t+i-1}}+1)^2+\phi_1(e^{2x_{t+i-1}}+1)^2(e^{2x_{t+i-1}}-1)+2\phi_1x_{t+i-1}e^{2x_{t+i-1}}(3e^{2x_{t+i-1}}-1)(e^{2x_{t+i-1}}+1)+\phi_2^2(2x_{t+i-2}e^{2x_{t+i-1}}(3e^{2x_{t+i-1}}+1)(e^{2x_{t+i-1}}-1))+\phi_3^3(6x_{t+i-3}e^{2x_{t+i-1}}(e^{2x_{t+i-1}}-1)^2)}{(e^{2x_{t+q-1}}+1)^3}]\zeta_t$$

$$+\prod_{i=2}^{q}[\frac{\phi_2^2(e^{2x_{t+i-1}}-1)^3}{(e^{2x_{t+i-1}}+1)^3}]\zeta_t+\prod_{i=3}^{q}[\frac{\phi_3^3(e^{2x_{t+i-1}}-1)^3}{(e^{2x_{t+i-1}}+1)^3}]\zeta_t \qquad (26)$$

Therefore, the limit cycle of the purposed model is orbitally stable if

$$\left|\prod_{i=1}^{q}[\frac{-6x_{t+i}e^{2x_{t+i-1}}(e^{2x_{t+i-1}}+1)^2+\phi_1(e^{2x_{t+i-1}}+1)^2(e^{2x_{t+i-1}}-1)+2\phi_1x_{t+i-1}e^{2x_{t+i-1}}(3e^{2x_{t+i-1}}-1)(e^{2x_{t+i-1}}+1)+\phi_2^2(2x_{t+i-2}e^{2x_{t+i-1}}(3e^{2x_{t+i-1}}+1)(e^{2x_{t+i-1}}-1))+\phi_3^3(6x_{t+i-3}e^{2x_{t+i-1}}(e^{2x_{t+i-1}}-1)^2)}{(e^{2x_{t+q-1}}+1)^3}]\right.$$

$$\left.+\prod_{i=2}^{q}[\phi_2^2\tanh^3 x_{t+i-1}]+\prod_{i=3}^{q}[\phi_3^3\tanh^3 x_{t+i-1}]\right|<1 \qquad (27)$$

4.Examples

In this section we give three examples to explain how to find the singular points of the proposed model, the conditions of stability of singular points and the limit cycle, and also we explain the following note.

Note: if the model has unstable singular point, then it has a stable limit cycle and the converse is also true.

Example(1):

In this example we show that, if the model have **unstable** singular point, then the model may have a **stable** limit cycle.

If $\phi_1=2$**,** then the model in equation (2) is

$$x_t=[2\tanh x_{t-1}]x_{t-1}+Z_t$$

The singular point:

By using equation(3), we get the non-zero singular point, which is

$$\zeta = \tanh^{-1}(\frac{1}{2}) = \tanh^{-1}(0.5) = 0.55$$

The stability of singular point:

Apply equation (6), we have that $\zeta_t = 1.825\zeta_{t-1}$ (28)

Then equation(28) is a first order linear autoregressive process.

Since the root $\lambda_1 = 1.825$ of the characteristic equation of equation (28) is lies outside the unit circle, then the singular point is not **stable**.

The limit cycle:

Let the limit cycle of period q=4 , which is $\{0.03, 0.13.0.26, 0.37, 0.03\}$

Then, from equation (11) we get:

$$\left| \prod_{i=1}^{4} [\frac{4x_{t+i-1}e^{2x_{t+i-1}} + 2(e^{2x_{t+i-1}} - 1) - 2e^{2x_{t+i-1}} x_{t+i})}{(e^{2x_{t+i-1}} + 1)}] \right| = 0.178 < 1$$

Therefore, the model has a **stable** limit cycle.

<u>Example(2):</u>

In this example we show that, if the model have a **stable** singular point, then the model maybe have **unstable** limit cycle.

If $\phi_1 = -0.9798$**,** then the model in equation (2) is

$$x_t = [-0.9798 \tanh x_{t-1}] x_{t-1} + Z_t$$

The singular point:

By using equation(3), we get the non-zero singular point, which is

$$\zeta = \tanh^{-1}(\frac{1}{-0.9798}) = \tanh^{-1}(-1.0206) = -2.2925 - 1.5708i$$

The stability of singular point:

Apply equation (6), we have that $\zeta_t = (0.90642 - 0.064121i)\zeta_{t-1}$ (29)

Then equation(29) is a first order linear autoregressive process.

Since the root $|\lambda_1| = 0.90869 < 1$ of the characteristic equation of equation (29) is lies inside the unit circle, then, the singular point is **stable**.

The limit cycle:

Let the limit cycle of period q=4 , which is

$\{-0.15, -0.4, -0.7, 1, -0.15\}$

Then, from equation (11) we get: $\left|\prod_{i=1}^{4}\left[\frac{4x_{t+i-1}e^{2x_{t+i-1}} + 2(e^{2x_{t+i-1}} - 1) - 2e^{2x_{t+i-1}}x_{t+i})}{(e^{2x_{t+i-1}} + 1)}\right]\right| = 16.8 > 1$

Therefore, the model has **unstable** limit cycle.

<u>Example(3):</u>

If $\phi_1 = -0.44444, \phi_2 = -0.63158$, then equation (12) becomes as

$x_t = [-0.44444 \tanh x_{t-1}]x_{t-1} + [(-0.63158)^2 \tanh^2 x_{t-1}]x_{t-2} + Z_t$

The singular point:

By using equation(13) we get two non-zero singular points, which are $\zeta_1 = 0.48133 + 1.5708\text{i}, \zeta_2 = -1.4304 - 1.5708i$

The stability of singular point:

If $\zeta_1 = 0.48133 + 1.5708\text{i}$

Then apply equation (14) we get
$\zeta_t = (-3.5703 - 8.409i)\zeta_{t-1} + (1.9936 + 4.366e - 016i)\zeta_{t-2}$ (30)

The characteristic equation of linear model of equation (30) is
$v^2 - (-3.5703 - 8.409i)v - (1.9936 + 4.366e - 016i) = 0$

Then $\lambda_1 = 0.090382 - 0.20261i, \lambda_2 = -3.6607 - 8.2064i$ are the roots of the characteristic equation of the model.

Then, the singular point is not **stable** because of one of the roots of the characteristic equation $|\lambda_2| = 8.9859$ lies outside the unit circle.

If $\zeta_2 = -1.4304 - 1.5708i$

Then we have :

$$\zeta_t = (0.0051628 - 0.54163i)\zeta_{t-1} + (0.50161 + 1.4106e - 017i)\zeta_{t-2} \quad (31)$$

The characteristic equation of linear model of equation (31) is
$v^2 - (0.0051628 - 0.54163i)v - (0.50161 + 1.4106e - 017i) = 0$

Then $\lambda_1 = 0.65701 - 0.27188i, \lambda_2 = -0.65185 - 0.26975i$ are the roots of the characteristic equation of the model in equation(31).

Since, all the roots of the characteristic equation are lies inside the unit circle, then the model have a **stable** singular point.

The limit cycle:

Let the limit cycle of period q=4 , which is

$\{-0.01, -0.1, -2, 2, -0.01\}$

Then, from equation (19) we get that

$$\left| \prod_{i=1}^{4} \left[\frac{-0.44444(4x_{t+i-1}e^{4x_{t+i-1}} + e^{4x_{t+i-1}} - 1) + 0.39(4x_{t+i-2}e^{2x_{t+i-1}}(e^{2x_{t+i-1}} - 1)) - 4x_{t+i}e^{2x_{t+i-1}}(e^{2x_{t+i-1}} + 1)}{(e^{2x_{t+i-1}} + 1)^2} \right] + \prod_{i=2}^{4} \left[\frac{0.39(e^{2x_{t+i-1}} - 1)^2}{(e^{2x_{t+i-1}} + 1)^2} \right] \right| = 0.016 + 0.000000064 = 0.016000064 < 1$$

Therefore, the model has a **stable** limit cycle.

5. Extract

1- We find the non-zero singular point of the proposed model of order one and two and three.

2- We find the stability conditions of the limit cycle of the proposed model of order one and two and three.

3- We explain the stability conditions of a non-zero singular point and the stability conditions of the limit cycle in three examples and find that the model of order one. Example(1) is not stable singular point and a stable limit cycle and find that the model of order one. Example(2) is stable singular point and unstable limit cycle and find that the model of order two. Example(3) have a two complex singular points ζ_1 , ζ_2 one of them ζ_1 is not stable and the other ζ_2 is stable, and the model is a stable limit cycle.

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