Three Near Points Method for Calculating Generalized Curvature and Torsion

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الملخص

الهدفَ من هذا البحث هو تأسيس طريقة جديدة لحِساب التقوسِ المعمـمِ والالتـواء المعمم. تَعتمد هذه الطريقةِ على الأقل على ثلاث نقاطِ متناهية القرب من النقطةِ المعتَبرةِ. إن الطريقةَ المؤَسسةَ تُميز عن الطرق الأخرى باستعمال عمليتي الضرب العـددي والاتجـاهي للمتجهات المكونة من تلك النقاط ، بدون استخدام أية كميات أو عناصر الداخلة فـي تعريـف المنحني وذلك باستخدام بعض مفاهيم التحليل غير القياسي الذي أعطتْ مِن قِبل Robinson, A. ووضعه Nelson, E. بأسلوب منطقي..

ABSTRACT

The aim of this paper is to establish a new method for calculating generalized curvature and generalized torsion. This method depends on at least three points infinitely close to the considered point. The established method is characterize from other methods by using only dot and cross product of vectors combined from this points, without any parameter that interacts in the defined curve by using some concepts of nonstandard analysis given by **Robinson A.** and axiomatized by **Nelson E.**

1- Introduction: -

In this paper the problem of obtaining the **generalized curvature**[6], [7], and **generalized torsion**[8], is considered and studied by using three points infinitely close to the given point. This method is different from the two methods given in [7], [8] and [9], and it has an advantage that it does not depend on any parameter which interact with the curve.

The following definitions and statements of nonstandard analysis will be needed throughout this paper. [3], [9], [10], [11], [14], [15], [16].

The axioms of IST (**Internal Set Theory**) which given by **Nelson E.** [11] are the axioms of ZFC (**Zermelo-Fraenckel Set Theory with Axiom of Choice**) together with three additional axioms, which are called the Transfer principle **(T)**, the Idealization principle **(I),** and the Standardization principle **(S)**, they are stated in the following

1- Transfer principle **(T)**:

Let $A(x,t_1,\ldots,t_k)$ be an internal formula with the free variables $x,t_1,\ldots,t_k$ only, then

$$\forall^{st} t_1 \cdots \forall^{st} t_k (\forall^{st} x \;\; A(x,t_1,\ldots,t_k) \Leftrightarrow \forall x(x,t_1,\ldots,t_k))$$

2- Idealization principle **(I)**:

Let $B(x,y)$ be an internal formula with the free variables x, y and with possibly other free variables. Then

$$\forall^{st\ fin} z \;\; \exists x \; \forall y \in z \;\; \wedge B(x,y) \Leftrightarrow \exists x \forall^{st} y \;\; \wedge B(x,y)$$

3- Standardization principle **(S)**:

Let $C(z)$ be a formula, internal or external, with the free variable z and with possibly other free variables. Then

$$\forall^{st} x \;\; \exists^{st} y \; \forall^{st} z (z \in y \Leftrightarrow z \in x \;\; \wedge C(z))$$

Every set or element defined in a classical mathematics is called **standard**.

Any set or formula which does not involve new predicates “standard, infinitesimals, limited, unlimited…etc” is called **internal,** otherwise it is called **external.**

A real number x is called **unlimited** if and only if $|x| > r$ for all positive standard real numbers, otherwise it is called **limited**.

The set of all unlimited real numbers is denoted by $\overline{\mathbf{R}}$, and the set of all limited real numbers is denoted by $\underline{\mathbf{R}}$

A real number x is called **infinitesimal** if and only if $|x| < r$ for all positive standard real numbers r .

A real number x is called **appreciable** if it is neither unlimited nor infinitesimal, and the set of all positive appreciable numbers is denoted by A^+**.**

Two real numbers x and y are said to be **infinitely close** if and only if $x - y$ is infinitesimal and denoted by $x \cong y$.

If x is a limited number in **R**, then it is infinitely close to a unique standard real number, this unique number is called the **standard part** of x or **shadow** of x denoted by $st(x)$ or ${}^0x$.

Necessary definitions in classical and non classical geometry can be found in [4], [5], and [12].

By a **parameterized differentiable curve**, we mean a differentiable map □:**I** □ $\mathbf{R}^3$ of an open interval **I** =(a,b) of the real line **R** in to **R** such that: □(t)=(x(t), y(t), z(t)) = $x(t)e_1$+ $y(t)e_2$+ $z(t)e_3$, and x, y, and z are differentiable at t; it is also called **spherical curve**.

Let □: **I** □ $\mathbf{R}^3$, **I** =(a,b), be a curve parameterized by an arc length s, and its tangent vector be □ ' which has a unit length, then the measure of the ratio of change of the angle with neighboring tangents made with the tangent at s is known as a **curvature** of the curve □ at s ,and it is given by $|g''(s)|$.

The **torsion** of the curve □: **I** □ $\mathbf{R}^3$ at s is define by $\dfrac{|\langle g'(t)\times g''(t), g'''(t)\rangle|}{|g'(t)\times g''(t)|^2}$

Theorem [6]

Let γ be a standard curve of order C^n and **A** be a standard singular point of order p-1 on γ; and let B and C be two points infernally close to the point A, then the generalized curvature of γ at the point A is given as follows and denoted by K_G

$$\frac{(p\,!)^{\frac{q}{p}}\left|x^{(p)}y^{(q)} - x^{(q)}y^{(p)}\right|}{q\,!\left(x^{(p)^2} + y^{(p)^2}\right)^{\frac{q+p}{2p}}} = \frac{(p\,!)^{\frac{q}{p}}\left|\gamma^{(p)} \times \gamma^{(q)}\right|}{q\,!\left\|\gamma^{(p)}\right\|^{\frac{q}{p}+1}},$$ where q is the order of the first vector derivative of γ not collinear with $\gamma^{(p)}$.

Theorem [8]

Let γ be a standard curve of order C^n and **A** be a standard singular point of **order** p-1 on γ; and let B and C be two points infernally close to the point A, then the generalized torsion of the curve γ is given as follows and denoted by τ_G.

$$ {}^{o}\tau_G = \frac{q!\cdot(p!)^{\frac{s-q}{p}}\left(\gamma^{(p)}(t_o)\times\gamma^{(q)}(t_o)\right)\bullet\gamma^{(s)}(t_o)}{s!\left\|\gamma^{(p)}\right\|^{\frac{s-q-p}{p}}\cdot\left(\gamma^{(p)}(t_o)\times\gamma^{(q)}(t_o)\right)^2} $$

where q is the order of the first vector derivative of γ not collinear with $\gamma^{(p)}$

2- Main Results:-

Theorem 2.1

Let γ be a standard curve of C^{∞} in $\mathbf{R}^3$. If B, C and D are three points infinitely close to the point $A=\gamma(t_o)$ such that the curve γ at A admitting at least three derivative vectors not coplanar then, the generalized torsion is given by

$$ \tau_G = \frac{\left|\left\langle \overrightarrow{AB}\times\overrightarrow{AC}, \overrightarrow{AD}\right\rangle\right|}{\left\|\overrightarrow{AC}\right\|^{\frac{q-2p}{2p}}\left\|\overrightarrow{AD}\right\|^{\frac{2p-3q}{2p}}\left[\left|\left\langle(\overrightarrow{AB}\times\overrightarrow{AC}),(\overrightarrow{AB}\times\overrightarrow{AD})\right\rangle\right|.\left|\left\langle(\overrightarrow{AC}\times\overrightarrow{AD}),(\overrightarrow{BC}\times\overrightarrow{BD})\right\rangle\right|\right]^{\frac{1}{2}}} $$

,

where p, q and s, $(p< q< s)$ are the first derivative orders such that $\left(\gamma^{(p)},\gamma^{(q)},\gamma^{(s)}\right)\neq 0$.

Proof:

By transfer (T) the curve γ and the point $A=\gamma(t_o)$ can be taken standards and let

$B=\gamma(t_o+\alpha\varepsilon\eta)$

$C=\gamma(t_o+\alpha\varepsilon)$

$D=\gamma(t_o+\alpha)$ with ${}^{o}\alpha={}^{o}\varepsilon={}^{o}\eta=0$

By using Taylor development to the order s we get:

$$ \overrightarrow{AB} = \alpha^p\varepsilon^p\eta^p\frac{\gamma^{(p)}(t_o)}{p!}+\cdots+\alpha^q\varepsilon^q\eta^q\frac{\gamma^{(q)}(t_o)}{q!}+\cdots+\alpha^s\varepsilon^s\eta^s\frac{\gamma^{(s)}(t_o)^s}{s!}+\delta_1\alpha^s\varepsilon^s\eta^s; \quad \cdots(2.1.1) $$

$$ \overrightarrow{AC} = \alpha^p\varepsilon^p\frac{\gamma^{(p)}(t_o)}{p!}+\cdots+\alpha^q\varepsilon^q\frac{\gamma^{(q)}(t_o)}{q!}+\cdots+\alpha^s\varepsilon^s\frac{\gamma^{(s)}(t_o)^s}{s!}+\delta_2\alpha^s\varepsilon^s; \quad \cdots(2.1.2) $$

$$\overrightarrow{AD} = \alpha^p \frac{\gamma^{(p)}(t_o)}{p!} + \cdots + \alpha^q \frac{\gamma^{(q)}(t_o)}{q!} + \cdots + \alpha^s \frac{\gamma^{(s)}(t_o)}{s!}^s + \delta_3 \alpha^s, \qquad \cdots(2.1.3)$$

thus

$$\left\|\overrightarrow{AC}\right\|^{\frac{q-2p}{2p}} = \left(\frac{\alpha^p}{p!}\right)^{\frac{q-2p}{2p}} \left\|\gamma^{(p)}(t_o) + \cdots + \alpha^{q-p}\varepsilon^{q-p} \frac{\gamma^{(q)}(t_o)}{(q-p)!} + \cdots + \alpha^{s-p}\varepsilon^{s-p} \frac{\gamma^{(s)}(t_o)}{(s-p)!}^s + \delta_3 \alpha^{s-p}\right\|^{\frac{q-2p}{2p}}$$

$$\cong \left(\frac{\alpha^p \varepsilon^p}{p!}\right)^{\frac{q-2p}{2p}} \left\|\gamma^{(p)}(t_o)\right\|^{\frac{q-2p}{2p}} \qquad \cdots(2.1.4)$$

and also

$$\left\|\overrightarrow{AD}\right\|^{\frac{2s-3q}{2p}} = \left(\frac{\alpha^p}{p!}\right)^{\frac{2s-3q}{2p}} \left\|\gamma^{(p)}(t_o) + \cdots + \alpha^{q-p} \frac{\gamma^{(q)}(t_o)}{(q-p)!} + \cdots + \alpha^{s-p} \frac{\gamma^{(s)}(t_o)}{(s-p)!}^s + \delta_3 \alpha^{s-p}\right\|^{\frac{2s-3q}{2p}}$$

$$\cong \left(\frac{\alpha^p}{p!}\right)^{\frac{2s-3q}{2p}} \left\|\gamma^{(p)}(t_o)\right\|^{\frac{2s-3q}{2p}} \qquad \cdots(2.1.5)$$

Now we put $\overrightarrow{W_1} = \overrightarrow{AB} \times \overrightarrow{AC}$,$\overrightarrow{W_2} = \overrightarrow{AB} \times \overrightarrow{AD}$,$\overrightarrow{W_3} = \overrightarrow{AD} \times \overrightarrow{AC}$ and $\overrightarrow{W_4} = \overrightarrow{BC} \times \overrightarrow{BD}$

then $$\overrightarrow{W_1} = \frac{\alpha^{p+q}\varepsilon^{p+q}}{p!q!} \eta^p \left(1 - \eta^{q-p}\right)\left(\gamma^p(t_o) \times \gamma^q(t_o) + inf\right);$$

$$\overrightarrow{W_2} = \frac{\alpha^{p+q}\varepsilon^{p}}{p!q!} \eta^p \left(1 - \eta^{q-p}\varepsilon^{q-p}\right)\left(\gamma^p(t_o) \times \gamma^q(t_o) + inf\right);$$

$$\overrightarrow{W_3} = \frac{\alpha^{p+q}\varepsilon^{p}}{p!q!} \left(\varepsilon^{q-p} - 1\right)\left(\gamma^p(t_o) \times \gamma^q(t_o) + inf\right)$$

and

$$\overrightarrow{W_4} = \frac{\alpha^{p+q}\varepsilon^{p}}{p!q!} \left(1 - \eta^p\right)\left(1 - \varepsilon^q \eta^q\right)\left(\gamma^p(t_o) \times \gamma^q(t_o) + inf\right),$$

thus

$$\left|\left\langle \overrightarrow{W_1}, \overrightarrow{W_2} \right\rangle\right| \cong \frac{\alpha^{2(p+q)}\varepsilon^{2p+q}}{(p!q!)^2} \eta^{2p} \left\|\gamma^p(t_o) \times \gamma^q(t_o)\right\|^2,$$

and

$$\left|\left\langle \overrightarrow{W_3}, \overrightarrow{W_4} \right\rangle\right| \cong \frac{\alpha^{2(p+q)}\varepsilon^{2p}}{(p!q!)^2} \left\|\gamma^p(t_o) \times \gamma^q(t_o)\right\|^2 .$$

Therefore

$$\left|\left\langle \overrightarrow{W_1},\overrightarrow{W_2}\right\rangle\right|\cdot\left|\left\langle \overrightarrow{W_3},\overrightarrow{W_4}\right\rangle\right| \cong \frac{\alpha^{4(p+q)}\varepsilon^{4p}\eta^{2p}\varepsilon^{p}}{(p!q!)^4}\left\|\gamma^{p}(t_o)\times\gamma^{q}(t_o)\right\|^4,$$

and

$$\left[\left|\left\langle \overrightarrow{W_1},\overrightarrow{W_2}\right\rangle\right|\cdot\left|\left\langle \overrightarrow{W_3},\overrightarrow{W_4}\right\rangle\right|\right]^{\frac{1}{2}} \cong \frac{\alpha^{2(p+q)}\varepsilon^{2p}\eta^{p}\varepsilon^{\frac{p}{2}}}{(p!q!)^4}\left\|\gamma^{p}(t_o)\times\gamma^{q}(t_o)\right\|^2$$

Therefore

$$\left\|\overrightarrow{AC}\right\|^{\frac{q-2p}{2p}}\left\|\overrightarrow{AD}\right\|^{\frac{2s-3q}{2p}}\left[\left|\left\langle \overrightarrow{W_1},\overrightarrow{W_2}\right\rangle\right|\cdot\left|\left\langle \overrightarrow{W_3},\overrightarrow{W_4}\right\rangle\right|\right]^{\frac{1}{2}} \cong \frac{\alpha^{s+q+p}\varepsilon^{p+q}\eta^{p}}{(p!)^{\frac{s-q}{p}+1}(q!)^2}\left\|\gamma^{p}(t_o)\right\|^{\frac{s-q-p}{p}}\left\|\gamma^{p}(t_o)\times\gamma^{q}(t_o)\right\|^2$$

,

and we have

$$\left|\left\langle \overrightarrow{AB}\times\overrightarrow{AC},\overrightarrow{AD}\right\rangle\right| = \frac{\alpha^{s+q+p}\varepsilon^{q+p}\eta^{p}}{p!q!s!}\left|\left(1-\eta^{q-p}\right)\left\langle\gamma^{p}(t_o)\times\gamma^{p}(t_o),\gamma^{s}(t_o)\right\rangle\right|$$

$$\cong \frac{\alpha^{s+q+p}\varepsilon^{q+p}\eta^{p}}{p!q!s!}\cdot\left|\left\langle\gamma^{p}(t_o)\times\gamma^{p}(t_o),\gamma^{s}(t_o)\right\rangle\right|$$

Hence

$$\frac{\left|\left\langle \overrightarrow{AB}\times\overrightarrow{AC},\overrightarrow{AD}\right\rangle\right|}{\left\|\overrightarrow{AC}\right\|^{\frac{q-2p}{2p}}\left\|\overrightarrow{AD}\right\|^{\frac{2p-3q}{2p}}\left[\left|\left\langle \overrightarrow{W_1},\overrightarrow{W_2}\right\rangle\right|\left|\left\langle \overrightarrow{W_3},\overrightarrow{W_4}\right\rangle\right|\right]^{\frac{1}{2}}} = \frac{q!\cdot(p!)^{\frac{s-q}{p}}\left|\left\langle\gamma^{(p)}(t_o)\times\gamma^{(q)}(t_o),\gamma^{(s)}(t_o)\right\rangle\right|}{s!\left\|\gamma^{(p)}\right\|^{\frac{s-q-p}{p}}\cdot\left\|\left(\gamma^{(p)}(t_o)\times\gamma^{(q)}(t_o)\right)\right\|^2}$$

That is

$$\tau_G = \frac{\left|\left\langle \overrightarrow{AB}\times\overrightarrow{AC},\overrightarrow{AD}\right\rangle\right|}{\left\|\overrightarrow{AC}\right\|^{\frac{q-2p}{2p}}\left\|\overrightarrow{AD}\right\|^{\frac{2p-3q}{2p}}\left[\left|\left\langle(\overrightarrow{AB}\times\overrightarrow{AC}),(\overrightarrow{AB}\times\overrightarrow{AD})\right\rangle\right|\cdot\left|\left\langle(\overrightarrow{AC}\times\overrightarrow{AD}),(\overrightarrow{BC}\times\overrightarrow{BD})\right\rangle\right|\right]^{\frac{1}{2}}}$$

The following theorem gives some equivalent formulas of the generalized curvature. Studies of other types curvatures can be found in [1], [2], and [13]

Theorem 2.2

Let γ be a C^{∞} standard curve in $\mathbf{R}^3$, and let B, C and D are three points infinitely close to the point $A=\gamma(t_o)$ such that the curve γ at A admitting at least two derivative vectors not colinear, then the following given formulas are equivalent:

1- $${}^{o}\kappa_G=\frac{(p!)^{\frac{q}{p}}\left|\gamma^{(p)}\times\gamma^{(q)}\right|}{q!\left\|\gamma^{(p)}\right\|^{\frac{q}{p}+1}}$$

2- $$\frac{\left|\left\langle\overrightarrow{AB}\times\overrightarrow{AC},\overrightarrow{AD}\times\overrightarrow{AC}\right\rangle\right|}{\left\|\overrightarrow{AB}\right\|\left\|\overrightarrow{BC}\times\overrightarrow{BD}\right\|\cdot\left\|\overrightarrow{BC}\right\|^{\frac{q}{p}}}$$

3- $$\frac{\left|\left\langle\overrightarrow{AB}\times\overrightarrow{AC},\overrightarrow{AD}\times\overrightarrow{AC}\right\rangle\right|}{\left\|\overrightarrow{AB}\right\|\cdot\left\|\overrightarrow{BC}\times\overrightarrow{BD}\right\|\cdot\left\|\overrightarrow{AC}\right\|^{\frac{q}{p}}}$$

4- $$\frac{\left|\left\langle\overrightarrow{AB}\times\overrightarrow{AC},\overrightarrow{BC}\times\overrightarrow{BD}\right\rangle\right|}{\left\|\overrightarrow{AB}\right\|\cdot\left\|\overrightarrow{AD}\times\overrightarrow{AC}\right\|\cdot\left\|\overrightarrow{AC}\right\|^{\frac{q}{p}}}$$

5- $$\frac{\left|\left\langle\overrightarrow{AB}\times\overrightarrow{AC},\overrightarrow{BC}\times\overrightarrow{BD}\right\rangle\right|}{\left\|\overrightarrow{AB}\right\|\left\|\overrightarrow{AD}\times\overrightarrow{AC}\right\|\cdot\left\|\overrightarrow{BC}\right\|^{\frac{q}{p}}}$$

6- $$\frac{\left\|\overrightarrow{BC}\times\overrightarrow{BD}\right\|}{\left\|\overrightarrow{BC}\right\|\cdot\left\|\overrightarrow{BD}\right\|^{\frac{q}{p}}}$$

7- $$\frac{\left\|\overrightarrow{AD}\times\overrightarrow{AC}\right\|}{\left\|\overrightarrow{AC}\right\|\cdot\left\|\overrightarrow{AD}\right\|^{\frac{q}{p}}}$$

8- $$\frac{\left\|\overrightarrow{BC}\times\overrightarrow{BD}\right\|}{\left\|\overrightarrow{AC}\right\|\cdot\left\|\overrightarrow{AD}\right\|^{\frac{q}{p}}}$$

Proof:

We can check the equivalency of all given formulas either by circular equivalency $(1\Rightarrow 2\Rightarrow\ldots\Rightarrow 8\Rightarrow 1)$ or by proving equivalent of each formula with the general curvature

Now we prove $1\Leftrightarrow 2$ and $1\Leftrightarrow 6$

$$\frac{\left|\left\langle\overrightarrow{AB}\times\overrightarrow{AC},\overrightarrow{AD}\times\overrightarrow{AC}\right\rangle\right|}{\left\|\overrightarrow{AB}\right\|\cdot\left\|\overrightarrow{BC}\times\overrightarrow{BD}\right\|\cdot\left\|\overrightarrow{BC}\right\|^{\frac{q}{p}}}>\frac{\left|\left\langle\frac{\alpha^{p+q}\varepsilon^{p+q}\eta^{p}}{p!q!}\left(\gamma^{p}(t_o)\times\gamma^{p}(t_o)\right),\frac{\alpha^{p+q}\varepsilon^{p}}{p!q!}\left(\gamma^{p}(t_o)\times\gamma^{p}(t_o)\right)\right\rangle\right|}{\frac{\alpha^{p+q}\varepsilon^{p}\eta^{p}}{p!}\left\|\gamma^{p}(t_o)\right\|\cdot\frac{\alpha^{p+q}\varepsilon^{p}}{p!q!}\left\|\gamma^{p}(t_o)\times\gamma^{p}(t_o)\right\|\cdot\left(\frac{\alpha^{p}\varepsilon^{p}}{p!}\right)^{\frac{q}{p}}\left\|\gamma^{p}(t_o)\right\|^{\frac{q}{p}}}$$

$$= \frac{\dfrac{\alpha^{2(p+q)}\varepsilon^{2p+q}\eta^{p}}{(p!q!)^{2}}\left\|\gamma^{p}(t_o)\times\gamma^{p}(t_o)\right\|^{2}}{\dfrac{\alpha^{2p+2q}\varepsilon^{2p+q}\eta^{p}}{(p!)^{\frac{q}{p}+2}q!}\left\|\gamma^{p}(t_o)\right\|^{\frac{q}{p}+1}\cdot\left\|\gamma^{p}(t_o)\times\gamma^{p}(t_o)\right\|}$$

$$= \frac{(p!)^{\frac{q}{p}}\left\|\gamma^{(p)}\times\gamma^{(q)}\right\|}{q!\left\|\gamma^{(p)}\right\|^{\frac{q}{p}+1}}$$

Thus

$$^{o}K_G = \frac{\left|\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)\cdot\left(\overrightarrow{AD}\times\overrightarrow{AC}\right)\right|}{\left\|\overrightarrow{AB}\right\|\cdot\left\|\overrightarrow{BC}\times\overrightarrow{BD}\right\|\cdot\left\|\overrightarrow{BC}\right\|^{\frac{q}{p}}}$$

Now let us prove the equivalency of the formula **6** with the first formula

$$\frac{\left\|\overrightarrow{BC}\times\overrightarrow{BD}\right\|}{\left\|\overrightarrow{BC}\right\|\cdot\left\|\overrightarrow{BD}\right\|^{\frac{q}{p}}} > \frac{\left|\dfrac{\alpha^{p+q}\varepsilon^{p}}{p!q!}(1+\eta^{p})(1-\varepsilon^{q}\eta^{q})\left(\gamma^{p}(t_o)\times\gamma^{p}(t_o)\right)\right|}{\dfrac{\alpha^{p}\varepsilon^{p}}{p!}\left\|\gamma^{p}(t_o)\right\|\cdot\left(\dfrac{\alpha^{p}}{p!}\right)^{\frac{q}{p}}\cdot\left\|\gamma^{p}(t_o)\right\|^{\frac{q}{p}}}$$

That is

$$\frac{\left\|\overrightarrow{BC}\times\overrightarrow{BD}\right\|}{\left\|\overrightarrow{BC}\right\|\cdot\left\|\overrightarrow{BD}\right\|^{\frac{q}{p}}} = \frac{\left|\dfrac{\alpha^{p+q}\varepsilon^{p}}{p!q!}(1+\eta^{p})(1-\varepsilon^{q}\eta^{q})\left(\gamma^{p}(t_o)\times\gamma^{p}(t_o)\right)\right|}{\dfrac{\alpha^{p}\varepsilon^{p}}{p!}\left\|\gamma^{p}(t_o)\right\|\cdot\left(\dfrac{\alpha^{p}}{p!}\right)^{\frac{q}{p}}\cdot\left\|\gamma^{p}(t_o)\right\|^{\frac{q}{p}}}$$

$$= \frac{(p!)^{\frac{q}{p}}\left\|\gamma^{(p)}\times\gamma^{(q)}\right\|}{q!\left\|\gamma^{(p)}\right\|^{\frac{q}{p}+1}}$$

Thus

$$^{o}K_G = \frac{\left\| \overrightarrow{BC} \times \overrightarrow{BD} \right\|}{\left\| \overrightarrow{BC} \right\| . \left\| \overrightarrow{BD} \right\|^{\frac{q}{p}}}$$

Similarly way we can prove that the other formulas are equivalent to $^{o}K_G$.

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