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Extension of Jeffery Prior Information

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ABSTRACT

We present in this paper Bays method by using Jeffery prior information and new Bayesian using extension of Jeffery prior information for estimating the parameter of exponential distribution and survival function . It has been shown from the computational results that the method which gives the best estimates is an extension of Jeffery prior information .

توسيع معلومات جفري القبلية

المستخلص

تم في هذا البحث تحديد التوزيع القبلي الذي اقترحه جفري. وطُبق على التوزيع الأسي ذي المعلمـــة الواحـــدة حيـــثُ أثبتـــت تجـــارب المحاكـــاة لتنويعة من المعلمات وحجوم العينات ان نتائج التحديد لتقدير المعلمة دالـــة البقاء كانت أكثر دقة .

1- Introduction

One of the most useful and widely exploited models is the exponential distribution. Epstein remarks that the exponential distribution plays an important role in life experiments as the part played by normal distribution in agricultural experiments . The most widely used is the one parameter exponential distribution with probability density function .

$$f(t) = \lambda \exp [- \lambda t]$$

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where $t \geq 0$, $\theta = \dfrac{1}{\lambda} > 0$ is the average or the mean life .

There are some properties of exponential distribution :

(1) Exponential distribution is the only continuous distribution with a constant hazard function .

(2) The exponential survival function is given by : S(t)=Pr[T>t] .

$$S(t) = \exp - \left[\int_{o}^{t} \lambda \, dt \right]$$

$$S(t) = \exp-(\lambda t)$$

(3) The exponential cumulative distribution function is :

$$F(t) = \Pr[T \leq t] = 1\text{-}\Pr[T > t] = 1\text{-}S(t) = 1\text{-}\exp(-\lambda t)$$

(4) If a human being survived t hours , then the probability of surviving an additional h hours is exactly the same as the probability of surviving h hours of a new time, or the probability that it will die during some period of time in the future is independent of its age , such that

$$pr(T > t + h \setminus T > t) = \frac{pr(T > t + h \cap T > t)}{pr(T > t)}$$

$$= pr(T>h)$$

This probability is called "forget fullness" or "memory less ness"

Thus the exponential distribution has memory less ness property .

2- Bayes Estimator

2-1 Bayes Estimator Using Jeffery Prior Information

Consider the one parameter exponential death time distribution

$$f(t;\theta) = \frac{1}{\theta}\exp(-\frac{t}{\theta}) \ , \theta > 0$$

We find Jeffery prior by : $g(\theta) \propto \sqrt{I(\theta)}$, where $\sqrt{I(\theta)} = \sqrt{\text{Fisher} \ldots \text{information}}$, and

$$I(\theta) = -\,nE\left(\frac{\partial^2 \ln f(t,\theta)}{\partial\theta^2}\right)$$

$$f(t;\theta) = \frac{1}{\theta}\exp(-\frac{t}{\theta})$$

$$\frac{\partial \ln f(t,\theta)}{\partial\theta} = -\frac{1}{\theta} + \frac{t}{\theta^2}$$

$$\frac{\partial^2 \ln f(t,\theta)}{\partial\theta^2} = +\frac{1}{\theta^2} - \frac{2t}{\theta^3}$$

$$E\left(\frac{\partial^2 \ln f(t,\theta)}{\partial\theta^2}\right) = \frac{1}{\theta^2} - \frac{2}{\theta^2} = \frac{-1}{\theta^2}$$

$$I(\theta) = -nE\left(\frac{\partial^2 \ln f(t,\theta)}{\partial\theta^2}\right) = \frac{n}{\theta^2}$$

when $g(\theta) \propto \sqrt{I(\theta)}$ $,\Rightarrow$ $g(\theta) \propto \frac{\sqrt{n}}{\theta}$

$g(\theta) = k\frac{\sqrt{n}}{\theta}$ where k is the normalizing factor

$$L(t_1\ldots,t_n \mid \theta) = \prod_{h=1}^{n} f(t_i \mid \theta)$$

$$= \frac{1}{\theta^{\;n}}\exp\left(-\frac{\sum_{i=1}^{n} t_i}{\theta}\right)$$

The joint probability density function $f(t_1,t_2,\ldots.t_n,\theta)$ is given by

$$H(t_1,....,t_n,\theta)=\prod_{i=1}^{n} f(t_i,\theta)g(\theta)=L(t_i,....,t_n\mid \theta)g(\theta)$$

$$=\frac{1}{\theta^n}\exp\left(-\frac{\sum_{i=1}^{n} t_i}{\theta}\right)\frac{k\sqrt{n}}{\theta}$$

$$=\frac{k\sqrt{n}}{\theta^{n+1}}\exp\left(-\frac{\sum_{i=1}^{n} t_1}{\theta}\right)$$

The marginal probability density function of $(t_1,….t_n)$ is given by
$P(t_1,…t_n) = \int H(t_1,….t_n,\theta)\, d\theta$

$$=\int_0^\infty \frac{k\sqrt{n}}{\theta^{n+1}}\exp\left(-\frac{\sum_{i=1}^{n} t_i}{\theta}\right)d\theta \qquad ,\text{let} \qquad y=\frac{\sum_{i=1}^{n} t_i}{\theta}$$

$$=\left(k\sqrt{n}\right)\int_0^\infty\left(\frac{\sum_{i=1}^{n} t_1}{y}\right)^{-n-1} \exp\ (-y)\left(\frac{\sum_{i=1}^{n} t_1}{y^2}\right)dy$$

$$=\left(k\sqrt{n}\right)\left(\sum_{i=1}^{n} t_i\right)^{-n}\int_0^\infty y^{n-1}\exp(-y)dy$$

$$=\frac{\left(k\sqrt{n}\right)\left(n-1\ \right)\ !}{\left(\sum_{i=1}^{n} t_1\right)^{n}}$$

and the conditional probability density function of θ given the data $(t_1,\ldots t_n)$ is given by : $\prod\left(\theta|t_1,\ldots,t_n\right)=\dfrac{H(t_1,\ldots,t_n,\theta)}{p(t_1,\ldots,t_n)}$

$$=\frac{\dfrac{k\sqrt{n}}{\theta^{n+1}}\exp\left(-\dfrac{\sum_{i=1}^{n}t_i}{\theta}\right)}{\dfrac{k\sqrt{n}}{\left(\sum_{i=1}^{n}t_i\right)^n}(n-1)!}$$

$$=\frac{\exp\left(-\dfrac{\sum_{i=1}^{n}t_i}{\theta}\right)}{\theta^{n+1}}\;\frac{\left(\sum_{i=1}^{n}t_i\right)^n}{(n-1)!}$$

using squared loss function $\ell\left(\hat{\theta},\theta\right)=c\left(\hat{\theta}-\theta\right)^2$

The risk function is $R\left(\hat{\theta},\theta\right)=E\ell\left(\hat{\theta},\theta\right)$

$$=\int_0^{\infty}\ell\left(\hat{\theta},\theta\right)\prod\left(\theta|t_1,\ldots,t_n\right)d\theta$$

$$=\int_0^{\infty}c\left(\hat{\theta}-\theta\right)^2\frac{\exp\left(-\dfrac{\sum_{i=1}^{n}t_i}{\theta}\right)\left(\sum_{i=1}^{n}t_i\right)^n}{\theta^{n+1}(n-1)!}d\theta$$

$$= c\hat{\theta}^2 - 2c\hat{\theta}\int_0^\infty \frac{\left(\sum_{i=1}^{n} t_i\right)^n}{(n-1)!}\theta^{-n} \exp\left(\frac{\sum_{i=1}^{n} t_i}{\theta}\right) d\theta + \psi(\theta)$$

$$\frac{\partial R(\hat{\theta},\theta)}{\partial\hat{\theta}} = 2c\hat{\theta} - 2c\frac{\left(\sum_{i=1}^{n} t_i\right)^n}{(n-1)!}\int_0^\infty \theta^{-n} \exp\left(-\frac{\sum_{i=1}^{i} t_i}{\theta}\right) d\theta + \text{zero}$$

Let $\frac{\partial R(\hat{\theta},\theta)}{\partial\hat{\theta}} = 0$, then

$$\hat{\theta}_B = \frac{\left(\sum_{i=1}^{n} t_i\right)^n}{(n-1)!}\int_0^\infty \theta^{-n} \exp\left(-\frac{\sum_{i=1}^{n} t_i}{\theta}\right) d\theta \qquad \text{let} \qquad y = \frac{\sum_{i=1}^{n} t_i}{\theta}$$

$$= \frac{\left(\sum_{i=1}^{n} t_i\right)^n}{(n-1)!}\int_0^\infty \left(-\frac{\sum_{i=1}^{n} t_i}{y}\right)^{-n} \exp(-y)\left(-\frac{\sum_{i=1}^{n} t_i}{y^2}\right) dy$$

$$= \frac{\left(\sum_{i=1}^{n} t_i\right)^n}{(n-1)!}\int_0^\infty y^{n-2} \exp(-y) dy$$

$$= \frac{\left(\sum_{i=1}^{n} t_i\right)(n-2)!}{(n-1)!}, \Rightarrow \hat{\theta}_B = \frac{\sum_{i=1}^{n} t_i}{n-1}$$

We can find the estimator of survival function by

$$\hat{s}_B(t)=\int_0^\infty \exp\left(\frac{-t_i}{\theta}\right)\prod(\theta|t_1,\ldots,t_n)d\theta$$

$$=\int_0^\infty \exp\left(\frac{-t_i}{\theta}\right)\frac{\exp\left(\frac{\sum_{i=1}^{n}t_i}{\theta}\right)\left(\sum_{i=1}^{n}t_i\right)^n}{\theta^{n+1}(n-1)!}d\theta$$

$$=\frac{\left(\sum_{i=1}^{n}t_i\right)^n}{(n-1)!}\int_0^\infty \theta^{-n-1}\exp\left(\frac{t_i-\sum_{i=1}^{n}t_i}{\theta}\right)d\theta \qquad \text{let} \qquad y=\frac{t_i-\sum_{i=1}^{n}t_i}{\theta}$$

$$=\frac{\left(\sum_{i=1}^{n}t_i\right)^n}{(n-1)!}\int_0^\infty \left(\frac{t_i+\sum_{i=1}^{n}t_i}{y}\right)^{-n-1}\exp(-y)\left(\frac{t_i+\sum_{i=1}^{n}t_i}{y^2}\right)dy$$

$$=\frac{\left(\sum_{i=1}^{n}t_i\right)^n}{(n-1)!}\left(t_i+\sum_{i=1}^{n}t_i\right)^{-n}(n-1)!$$

$$=\frac{\left(\sum_{i=1}^{n}t_i\right)^n}{\left(t_i+\sum_{i=1}^{n}t_i\right)^n}=\left(1+\frac{t_i}{n\bar{t}}\right)^{-n}$$

2-2 Bayes Estimator using Extension of Jeffery

The extension of Jeffery prior is

$$g(\theta)\propto\left[I(\theta)\right]^{C1} \qquad C_1\in R^+$$

$$g(\theta) \propto \left[\frac{n}{\theta^2}\right]^{C1}$$

$$g(\theta) = k\frac{n^{C1}}{\theta^{2C_1}}$$

$$L(t_1,....t_n|\ \theta) = \prod_{i=1}^{n} f(t_i|\ \theta)$$

$$= \frac{1}{\theta^n}\exp\left(-\frac{\sum_{i=1}^{n} t_i}{\theta}\right)$$

The joint probability density function is given by

$$H(t_1,....t_n\theta) = \prod_{i=1}^{n} f(t_i \setminus \theta)g(\theta)$$

$$= \frac{1}{\theta^n}\exp\left(-\frac{\sum_{i=1}^{n} t_i}{\theta}\right)\frac{kn^{C1}}{\theta^{2C1}}$$

$$\frac{kn^{C_1}}{\theta^{n+2C_1}}\exp\left(\frac{\sum_{i=1}^{n} t_i}{\theta}\right)$$

The marginal probability density function of ($t_1,\ldots,tn$) is given by

$$P(t_1,\ldots,t_n) = \int_0^{\infty} H(t_1,...t_n,\theta)d\theta$$

$$= \int_0^{\infty} \frac{kn^{C_1}}{\theta^{n+2C_1}}\exp\left(-\frac{\sum_{i=1}^{n} t_i}{\theta}\right)d\theta \quad , \quad \text{let} \quad y = \frac{\sum_{i=1}^{n} t_i}{\theta}$$

$$= kn^{C_1} \int_0^\infty \left(\frac{\sum t_i}{y}\right)^{-n-2C_1} \exp(-y) \left(\frac{\sum_{i=1}^{n} t_i}{y^2}\right) dy$$

$$= \frac{kn^{C_1}}{\left(\sum_{i=1}^{n} t_i\right)^{n+2C_1-1}} \int_0^\infty y^{n+2C_1-2} \exp(-y) dy \qquad = \frac{kn^{C_1}(n+2C_1-2)!}{(\sum_{i=1}^{n} t_i)^{n+2C_1-1}}$$

The conditional probability density function of θ given the data $(t_1,\ldots t_n)$ is given by :

$$\frac{\partial R(\hat{\theta},\theta)}{\partial \theta} = 2c\hat{\theta} \;\; - 2c\int_0^\infty \frac{\theta^{-n-2C_1+1} \exp\left(-\frac{\sum_{i=1}^{n} t_i}{\theta}\right)}{\left(\sum_{i=1}^{n} t_i\right)^{1-n-2C_1} (n+2C-2)!} d\theta + \text{zero}$$

$$\prod(\theta \quad |t_1,\ldots,t_n) = \frac{\dfrac{\frac{kn^C}{n+2C_1} \exp\left(-\frac{\sum_{i=1}^{n} t_i}{\theta}\right)}{\theta}}{\dfrac{kn^{\;\;C_1}(n+2C_1-2)!}{\left(\sum_{i=1}^{n} t_i\right)}}$$

$$= \frac{\theta^{-n-2c1} \exp\left(-\frac{\sum_{i=1}^{n} t_i}{\theta}\right)}{\left(\sum_{i=1}^{n} t_i\right)^{1-n-2C} (n+2C-2)!}$$

Now, consider a squared error loss function , we can find Risk function

$$R(\hat{\theta},\theta) = \int_0^\infty C(\hat{\theta}-\theta)^2 \frac{\theta^{-n-2c1}\exp\left(-\frac{\sum_{i=1}^{n} t_i}{\theta}\right)}{\left(\sum_{i=1}^{n} t_i\right)^{1-n-2C_1}(n+2C-2)!} d\theta$$

$$= c\hat{\theta}^2 - 2c\hat{\theta}\int_0^\infty \frac{\theta^{-n-2C_1+1}\exp\left(-\frac{\sum_{i=1}^{n} t_i}{\theta}\right)}{\left(\sum_{i=1}^{n} t_i\right)^{1-n-2C_1}(n+2C-2)!} d\theta + \psi(\theta)$$

$$\text{let } \frac{\partial R(\hat{\theta},\theta)}{\partial\theta} = 0$$

$$\hat{\theta}_B = \frac{1}{\left(\sum_{i=1}^{n} t_i\right)^{1-n-2C_1}(n+2C_1-2)!}\int_0^\infty \theta^{1-n-2C_1}\exp\left(-\frac{\sum_{i=1}^{n} t_i}{\theta}\right)d\theta$$

$$= \frac{1}{\left(\sum_{i=1}^{n} t_i\right)^{1-n-2C_1}(n+2C_1-2)!}\int_0^\infty \left(\frac{\sum_{i=1}^{n} t_i}{y}\right)^{1-n-2C_1}\exp(-y)\left(\frac{\sum_{i=1}^{n} t_i}{y^2}\right)dy$$

$$= \frac{\left(\sum_{i=1}^{n} t_i\right)}{(n+2C_1-2)!}\int_0^\infty y^{n+2C1-3}\exp(-y)dy$$

$$= \frac{\left(\sum_{i=1}^{n} t_i\right)(n + 2C_1 - 3)!}{(n + 2C_1 - 2)!}$$

$$\hat{\theta}_B = \frac{\sum_{i=1}^{n} t_i}{n + 2C_1 - 2}$$

Now , when $C = \frac{1}{2}$, we will get

$$\hat{\theta}_B = \frac{\sum_{i=1}^{n} t_i}{n - 1}$$

Which is Jeffery’s estimator and it is a special case of proposed one . We can find estimation of survival function .

$$= \frac{1}{\left(\sum_{i=1}^{n} t_i\right)^{1-n-2C_1} (n + 2C - 2)!} \int_0^\infty \theta^{-n-2C_1} \exp\left(\frac{-t_i - \sum_{i=1}^{n} t_i}{\theta}\right) d\theta$$

$$\hat{s}_B(t) = \int_0^\infty \exp\left(-\frac{t_i}{\theta}\right) \prod (\theta | t_1, \ldots, t_n) d\theta$$

$$= \int_0^\infty \exp\left(-\frac{t_1}{\theta}\right) \frac{\theta^{-n-2C_1} \exp\left(-\frac{\sum_{i=1}^{n} t_i}{\theta}\right)}{\left(\sum_{i=1}^{n} t_i\right)^{1-n-2C_1} (n + 2C - 2)!} d\theta$$

$$= \frac{1}{\left(\sum_{i=1}^{n} t_i\right)^{1-n-2C_1} (n+2C_1-2)!} \int_0^\infty \left(\frac{t_1+\sum_{i=1}^{n} t_i}{y}\right)^{-n-2C_1} \exp(-y) \left(\frac{t_1+\sum_{i=1}^{n} t_i}{y^2}\right) d\theta$$

$$= \frac{\left(t_1+\sum_{i=1}^{n} t_i\right)^{1-n-2C_1}}{\left(\sum_{i=1}^{n} t_i\right)^{1-n-2C_1} (n+2C_1-2)!} \int_0^\infty y^{n+2C_1-2} \exp(-y) dy$$

$$\hat{S}_B(t) = \frac{\left(t_1+\sum_{i=1}^{n} t_i\right)^{1-n-2C_1} (n+2C_1-2)!}{\left(\sum_{i=1}^{n} t_i\right)^{1-n-2C_1} (n+2C_1-2)!}$$

$$= \left(1+\frac{t_i}{n\bar{t}}\right)^{1-2C_1-n}$$

$$= \left(\frac{t_i+\sum_{i=1}^{n} t_i}{\sum_{i=1}^{n} t_i}\right)^{1-n-2C_1}$$

3-Simulation

The simulation program is written by using Matlab program. This experiment chooses the samples with small size , moderate and large, namely , n = 25 , 50 , 100 , with varieties of parameter value , namely, θ =0.5, 1, 1.5 , 2 .

The size of replication is 1000 . To compare between the methods of estimator , uses MSE criterion .

$$\text{MSE} \quad (\hat{\theta}) = \frac{\sum_{i=1}^{R} (\hat{\theta} - \theta)^2}{R}$$

Table (1) : MSE of estimated parameter of exponential distribution

SIZE	B	c1	Bayes	PROPOSED	BEST
25	0.5	0.2	0.0115	0.0128	M1
		0.4	0.0110	0.0114	M1
		1	0.0114	0.0100	M2
		1.4	0.0110	0.0095	M2
	1	0.2	0.0415	0.0415	M1
		0.4	0.0478	0.0495	M1
		1	0.0436	0.0384	M2
		1.4	0.0436	0.0377	M2
	1.5	0.2	0.0979	0.1087	M1
		0.4	0.0960	0.0992	M1
		1	0.0986	0.0889	M2
		1.4	0.0913	0.0778	M2
	2	0.2	0.1874	0.2091	M1
		0.4	0.1895	0.1960	M1
		1	0.1740	0.1530	M2
		1.4	0.1795	0.1501	M2
50	0.5	0.2	0.0053	0.0056	M1
		0.4	0.0055	0.0056	M1
		1	0.0052	0.0048	M2
		1.4	0.0049	0.0047	M2
	1	0.2	0.0221	0.0235	M1
		0.4	0.0211	0.0214	M1
		1	0.0203	0.0193	M2
		1.4	0.0214	0.0202	M2
	1.5	0.2	0.0433	0.0456	M1
		0.4	0.0458	0.0465	M1
		1	0.0486	0.0460	M2
		1.4	0.0488	0.0447	M2
	2	0.2	0.0797	0.0893	M1
		0.4	0.0881	0.0896	M1
		1	0.0798	0.0748	M2
		1.4	0.0854	0.0803	M2
100	0.5	0.2	0.0026	0.0027	M1
		0.4	0.0025	0.0025	M1&M2
		1	0.0025	0.0024	M2
		1.4	0.0026	0.0025	M2
	1	0.2	0.110	0.113	M1

SIZE	B	c1	Bayes	PROPOSED	BEST
		0.4	0.0106	0.0106	M1&M2
		1	0.0103	0.0100	M2
		1.4	0.0101	0.0097	M2
	1.5	0.2	0.0236	0.0244	M1
		0.4	0.0221	0.0223	M1
		1	0.0234	0.0226	M2
		1.4	0.0219	0.0210	M2
	2	0.2	0.0384	0.0397	M1
		0.4	0.0394	0.0396	M1
		1	0.0426	0.0412	M2
		1.4	0.0406	0.0391	M2
Bayes					45.83%
Extention of Jeffery					50%
The same					3.166%

4 - Conclusion

1. The standard Bayes estimator is a special case of proposed extension of Jeffery prior information .
2. The extension of jeffery prior with squared error loss function is better than Bayes estimator .

5 – References

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