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Characterizing Internal and External Sets

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"تمييز المجموعات الداخلية والخارجية"

الملخص

الهدف من هذا البحث هو إعطاء تمييز بين المجموعات الخارجية والمجموعات الداخلية وبعض العلاقات بين الكالكسيات والهالات ، ومن أهم النتائج التي حصلنا عليها :

- تكون المجموعة G كالكسي **إذا وفقط إذا** وجدت متتابعة متزايدة بدقة من المجموعات الداخلية $\{T_n\}_{n\in\underline{\mathbb{N}}}$ بحيث أن $G=\bigcup_{n\in\underline{\mathbb{N}}} T_n$.

  كذلك تكون المجموعة H هالة **إذا وفقط إذا** وجدت متتابعة متناقصة بدقة من المجموعات الداخلية $\{S_n\}_{n\in\underline{\mathbb{N}}}$ بحيث أن $H=\bigcap_{n\in\underline{\mathbb{N}}} S_n$
- إذا كانت H هالة و G كالكسي بحيث $G\subset H$، فانه توجد مجموعة داخلية I بحيث أن $G\subset I\subset H$.
- إذا كانت H هالة و G كالكسي فان $G\neq H$. (أي أن الهالة لا تكون كالكسي).
- إذا كانت H هالة و G كالكسي فإن المجموعة G^H لكل الدوال الداخلية $f:G\to H$ بحيث أن $f(H)=G$ هالة.

ABSTRACT

The aim of this paper is to give a characterization between the external and internal sets, and some relation between the galaxies and monads, according to this paper we obtain the following results :

- A set G is galaxy **iff** there exists a strictly increasing sequence of internal sets $\{T_n\}_{n\in\underline{\mathbb{N}}}$ such that $G=\bigcup_{n\in\underline{\mathbb{N}}} T_n$.

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Also A set H is monad **iff** there exists a strictly decreasing sequence of internal sets $\{S_n\}_{n\in\underline{\mathbb{N}}}$ such that $H = \bigcap_{n\in\underline{\mathbb{N}}} S_n$.

- ❖ If G is a galaxy and H is a monad such that $G \subset H$, then there exists an internal set I such that $G \subset I \subset H$.
- ❖ A monad is not galaxy.
- ❖ If H is a monad and G is a galaxy, the set of all internal functions $f : G \to H$ such that $f(H) = G$ is a monad)

Keywords: Galaxy, Monad , Internal, External.

1. Introduction

The following definitions and notations are needed throughout this paper :

Every concept concerning sets or elements defined in the classical mathematics is called **standard**

Any set or formula which does not involve new predicates “standard, infinitesimals, limited, unlimited…etc” is called **internal,** otherwise it is called **external** [3,5].

A real number x is called **unlimited** if and only if $|x| > r$ for all positive standard real numbers r ; otherwise it is called **limited**.

The notations R, $\overline{R}$ and $\underline{R}$ denote respectively the set of real numbers, the set of all **unlimited** real numbers and the set of all **limited** real numbers.

A real number x is called **infinitesimal** if $|x| < r$ for all positive standard real numbers r

A real number x is called **appreciable** , if x is limited but not infinitesimal.

Two real numbers x and y are said to be **infinitely close** if and only if $x - y$ is infinitesimal and denoted by $x \simeq y$. [2,4,6,7]

The external set of infinitesimal real numbers is called the **monad of 0** (denoted by $m\ (0)$). In general, the set of all real numbers, which are infinitely close to a standard real number a ,is called the **monad of** a, (denoted by $m(a)$)

The set of all limited real numbers is called **principal galaxy**, (denoted by $\mathbb{G}$).

For any real number a, the set of all real numbers x such that $x - a$ limited is called the **galaxy of *a*** (denoted by $G\ (a)$).

Let α, $(\alpha \neq 0)$ and $x \in \mathbb{R}$, we define the $\alpha - galaxy(x)$ as follows:

$\alpha - galaxy(x) = \{y \in \mathbb{R} : \frac{y-x}{\alpha}\ is\ limited\}$, and denoted by $\alpha - G(x)$ [1,4].

Definition 1.1[3, 6] **:** A set G is called galaxy if

(i) G is an external set.

(ii) there is an internal sequence $\{A_n\}_{n\in\underline{\mathbb{N}}}$ of internal sets such that $G = \bigcup_{n\in\underline{\mathbb{N}}} A_n$.

A set H is called monad if

(i) H is an external set.

(ii) there is an internal sequence $\{B_n\}_{n\in\underline{\mathbb{N}}}$ of internal sets such that $H = \bigcap_{n\in\underline{\mathbb{N}}} B_n$.

Theorem 1.2: (Cauchy Principle) [7]

If p is any internal property and if $p(n)$ holds for all standard $n \in \mathbb{N}$, then there exists an unlimited $\omega \in \mathbb{N}$ such that $p(n)$ hold for all $n \leq \omega$.

proposition 1.3 :

(i) If $\{G_n\}_{n\in\underline{\mathbb{N}}}$ is a sequence of galaxies then $\bigcup_{n\in\underline{\mathbb{N}}} G_n$ is a galaxy.

(ii) If $\{H_n\}_{n\in\underline{\mathbb{N}}}$ is a sequence of monads then $\bigcap_{n\in\underline{\mathbb{N}}} H_n$ is a monad.

Proofs : follows directly from their definitions.

proposition 1.4 :

(i) The image and inverse image of a galaxy under internal mapping are galaxies.

(ii) The image and inverse image of a monad under internal mapping are monads.

Proofs : follows directly from their definitions of inverse functions.

Thus we consider the following theorem:

Theorem 1.5 :

(i) A set G is galaxy **iff** there exists a strictly increasing sequence of internal sets $\{T_n\}_{n\in\underline{\mathbb{N}}}$ such that $G = \bigcup_{n\in\underline{\mathbb{N}}} T_n$.

(ii) A set H is monad **iff** there exists a strictly decreasing sequence of internal sets $\{S_n\}_{n\in\underline{\mathbb{N}}}$ such that $H = \bigcap_{n\in\underline{\mathbb{N}}} S_n$.

Proof : We prove only (i)

Let G be a galaxy and let $\{T_k\}_{k\in\underline{\mathbb{N}}}$ be an internal sequence of internal sets such that $G = \bigcup_{k\in\underline{\mathbb{N}}} T_k$. We define a strictly increasing subsequence $\{I_n\}_{n\in\underline{\mathbb{N}}}$ of the sequence $\{T_k\}_{k\in\underline{\mathbb{N}}}$. We remark first $T_k \subsetneq G$; for all standard k , for G is external. So there exists for all standard k a standard natural number $p > k$ such that $T_k \subsetneq T_p$. Hence there exists by induction a strictly increasing sequence of standard natural numbers $\{K_n\}_{n\in\underline{N}}$ such that $T_{k_n} \subsetneq T_{k_{n+1}}$, $n \in \underline{\mathbb{N}}$. Putting $I_k \subsetneq T_{k_n}$, we obtain a strictly increasing sequence of internal sets $\{I_n\}_{n\in\underline{\mathbb{N}}}$ such that $G = \cup_{n\in\underline{N}} I_n$.

Conversely let $\{I_n\}_{n\in\underline{N}}$ be a strictly increasing sequence of internal sets. That is we may Putting $G = \cup_{n\in\underline{N}} I_n$. By the principle of extension, there exists an internal extension $\{I_n\}_{n\in N}$ of this sequence, that is we may assume it an increasing. Suppose G is internal set, since $I_n \subsetneq G$, for all $n \in \underline{\mathbb{N}}$, so there exists by Cauchy principle $\omega \in \overline{\mathbb{N}}$ such that $I_\omega \subsetneq G$. Therefore that we may assume it $\cup_{n\in\underline{N}} I_n \subsetneq G$ $\bigcup_{n\in\underline{\mathbb{N}}} T_n \subsetneq G$ which is a contradiction . Hence G is an external set, thus G is a galaxy ■.

Remark 1.6: Let X and Y be two internal sets and $f: X \longrightarrow Y$ an internal mapping , and $G_1 \subset X$ and $G_2 \subset Y$ are two galaxies, then

1- If $f\big|_{G_1}$ is one to one , then $f(G_1)$ is a galaxy.

2- If $f : X \to G_2$ is onto, then $f^{-1}(G_2)$ is a galaxy.

Theorem 1.7 :

(i) A subset G of an internal set X is a galaxy **iff** G is the inverse image of $\underline{\mathbb{N}}$ under an internal mapping from X in to $\mathbb{N}$.

(ii) A subset H of an internal set X is a monad **iff** H is the inverse image of $\underline{\mathbb{N}}$ under an internal mapping from X in to $\mathbb{N}$.

Proof :

(i) Let $G \subset X$ be a galaxy and let $\{T_n\}_{n\in\underline{\mathbb{N}}}$ be an internal increasing sequence of internal sets such that $G = \bigcup_{n\in\underline{\mathbb{N}}} T_n$. We may assume that $\bigcup_{n\in\underline{\mathbb{N}}} T_n = X$, we define the internal mapping $p: X \longrightarrow \mathbb{N}$ by $p(x) = min\,(n \in \mathbb{N}: x \in t_n\}$. Clearly we have $G = p^{-1}(\mathbb{N})$.

Conversely if $p^{-1}(\mathbb{N})$ is a galaxy for every internal mapping $p: X \longrightarrow \mathbb{N}$ by the **proposition (1.4)**■. $P(x) = \min\{\ldots, x \in T_{m_n}\}$

(ii) Let $H \subset X$ be a monad , putting $G = X - H$, let $p: X \longrightarrow \mathbb{N}$ be an internal mapping such that $G = p^{-1}(\mathbb{N})$, then we have $H = p^{-1}(\mathbb{N})$.

Conversely if $p^{-1}(\mathbb{N})$ is a monad for every internal mapping $p: X \longrightarrow \mathbb{N}$ again by the proposition (1.4) we get $H = p^{-1}(\mathbb{N})$■.

The converse follow directly from proposition (1, 4)

Proposition 1.8 : If G is a galaxy and H is a monad such that $G \subset H$, then there exists an internal set I such that $G \subset I \subset H$.

Proof : Let $\{T_n\}_{n\in\underline{\mathbb{N}}}$ be an internal increasing sequence of internal sets such that $G = \bigcup_{n\in\underline{\mathbb{N}}} T_n$ and let $\{K_n\}_{n\in\underline{\mathbb{N}}}$ be an internal decreasing sequence of internal sets such that $H = \bigcap_{n\in\underline{\mathbb{N}}} K_n$.

Since $T_n \subset K_n$ for all $n \in \underline{\mathbb{N}}$, there exists by Cauchy principle unlimited real number ω , such that $T_n \subset K_n$ for all natural number $n \leq \omega$ therefore

$$G = \bigcup_{n\in \underline{N}} T_n \subset \bigcup_{n\leq\omega} T_n = T_\omega \subset K_\omega = \bigcap_{n\leq\omega} K_n \subset \bigcap_{n\leq \underline{N}} K_n = H$$

Putting **for example** $I = T_\omega$, we obtain $G \subset I \subset H$.

Theorem 1.9 : (Fehrel Theorem) [2]

A monad is not galaxy.

Proof : Let G be a galaxy and H be a monad assume that $G \subset H$ by Proposition (1.8) we may let I be an internal set such that $G \subset I$, $I \subset H$ by Cauchy principle an external set is not internal $G \subsetneq I$, $I \subsetneq H$. Hence $G \neq H$ ∎.

2. Some Application of Fehrel Theorem

Robinson's Lemma 2.1[5]: If $\{a_n\}_{n\in\underline{\mathbb{N}}}$ is an internal sequence of real numbers such that $a_n \simeq 0$, for all $n \in \underline{\mathbb{N}}$, then there exists an unlimited natural number $\omega \in \mathbb{N}$ such that $a_n \simeq 0$, for all $n \leq \omega$.

Proof: Let b_k be the maximum $n \leq k|a_n|$ then also $b_k \simeq 0$, for all $n \in \underline{\mathbb{N}}$

Now by fehrel theorem the galaxy $\underline{\mathbb{N}}$ is a strictly included in the monad k such that $b_k \simeq 0$, the set of all k such that $b_k \simeq 0$. So there exists an unlimited $\omega \in \mathbb{N}$ such that $b_\omega \simeq 0$, hence $a_n \simeq 0$, for all $n \leq \omega$ ∎.

3. Functions From a Monad to Galaxy

Now we are able to prove the statement of the following form: The set of internal functions from a monad to galaxy is a galaxy.

Let G be a galaxy and H be a monad. Let $\{A\}_n$ $n \in N$ be an internal increasing sequence such that $G = \bigcup_{n\in\underline{N}} A_n$ and $\{B\}_n$ $n \in N$ be an internal decreasing sequence such that $H = \bigcap_{n\in\underline{N}} B_n$ then the set H^G of all internal mapping $f : G \to H$ such that $f(G) \subset H$ is a monad.
For
$f(G) \subset H \Leftrightarrow (\forall n \in \underline{N})(\forall m \in \underline{N})(f(A_n) \subset B_m)$
As may be expected G^H is a galaxy we prove this with the help of Fehrele's principle

Proposition 3.1 : If H is a monad and G is a galaxy then G^H is a galaxy.

Proof : Let $\{T_k\}_{k\in\underline{\mathbb{N}}}$ be an internal increasing sequence of internal set such that $G = \cup_{n\in\underline{N}} T_n$ and let $\{I_n\}_{n\in\underline{\mathbb{N}}}$ be an internal decreasing sequence of internal set such that $H = \bigcap_{n\in\underline{\mathbb{N}}} I_n$. We are going to prove that

$G^H = \cup_{m \in \underline{N}} \cup_{n \in \underline{N}} T_n^{I_{nm}}$. Clearly if $f(I_m) = T_n$ for some internal function and $n, m \in \underline{\mathbb{N}}$, then $f(H) \subset G$. Also let f be an internal function such that $f(H) \subset G$, put $m_n = min\{m: f(I_m) \subset T_n\}$, for every $n \in \mathbb{N}$, so $\{m_n\}_{n \in \underline{\mathbb{N}}}$ is internal sequence of natural numbers. Now suppose that $m_n \in \mathbb{N}$, for all $n \in \underline{\mathbb{N}}$. Then there exists by fehrel theorem $\omega \in \overline{\mathbb{N}}$ such that $m_\omega \in \overline{N}$ $m_n \in \overline{\mathbb{N}}$, so there exists $x \in I_{m_\omega - 1}$ such that $f(x) \notin T_\omega$. Hence $x \in H$, and $f(x) \notin G$, contradiction. So there exists $n \in \underline{\mathbb{N}}$ such that $m_n \in \underline{N}$. This implies that $f(I_{m_n}) \subset T_n$. Hence $G^H = \cup_{m \in \underline{N}} \cup_{n \in \underline{N}} T_n^{I_{nm}}$. So by proposition 1.3 G^H is a galaxy ■.

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